Mini-presentation on Bowen Notebook Problem 85

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Some Systems with Unique Equilibrium States

by

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We shall be dealing with a homeomorphism $f: X \to X$ of a compact metric space and a continuous $\varphi: X \to \mathbb{R}$. Let $M_f(X)$ denote the set of all $f$-invariant Borel probability measures on $X$. $\mu \in M_f(X)$ is called an equilibrium state (for $f$ and $\varphi$) if

$$h_\mu(f) + \mu(\varphi) = \sup_{\nu \in M_f(X)} (h_\nu(f) + \nu(\varphi)),$$

where $h_\mu(f)$ is the entropy of $\mu$. We want conditions on $f$ and $\varphi$ which guarantee a unique equilibrium state.

$f$ is called expansive if there is an $\epsilon > 0$ such that for any two points $x \neq y$ in $X$ there is an $n \in \mathbb{Z}$ so that $d(f^n(x), f^n(y)) > \epsilon$ satisfies specification if for each $\delta > 0$ there is an integer $n(\delta)$ for which the following is true: if $l_1, \cdots, l_n$ are intervals of integers contained in $[a, b]$ with $d(l_i, l_j) \geq \delta$ for $i \neq j$ and $x_1, \cdots, x_n \in X$, then there is a point $x \in X$ with $f^{n(\delta)}(x) = x$ and $d(f^n(x), f^n(x_i)) < \delta$ for $k \in l_i$. This condition allows us to construct a lot of periodic points.

For $\varphi \in C(X)$ and $n \geq 1$ let

$$(S_\varphi^n)(x) = \varphi(x) + \varphi(f(x)) + \cdots + \varphi(f^{n-1}(x)).$$

Let $V(f)$ be the set of $\varphi \in C(X)$ for which an $\epsilon > 0$ and a $K$ exist for which the following is true: $d(f^n(x), f^n(y)) \leq \epsilon$ for all $0 \leq k < n \Rightarrow |S_\varphi^n(x) - S_\varphi^n(y)| \leq K$.

**Theorem.** Let $f: X \to X$ be an expansive homeomorphism of a compact metric space satisfying specification. Then each $\varphi \in V(f)$ has a unique equilibrium state $\mu_\varphi$.

**Remark.** Let $\delta$ be any expansive constant for $f$. Then, if $\varphi \in V(f)$, $|\varphi|_f = \sup (|S_\varphi^n(x) - S_\varphi^n(y)| : n \geq 1 \text{ and } d(f^n(x), f^n(y)) \leq \delta \forall k \in [0, n])$ is finite (if $\epsilon, K$ are as in the definition of $\varphi \in V(f)$ and $d(f^n(x), f^n(y)) \leq \delta$ follows from $d(f^n(x), f^n(y)) \leq \epsilon$).
Question (85.)

Codon frequencies via equilibrium states for “some potential”?
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Recent progress:


- Also papers by Bruno Cessac e.g. *Gibbs distribution analysis of temporal correlations structure in retina ganglion cells*, Journal of Physiology, 2012: Uses topological pressure in neural networks; cites Bowen, Ruelle, etc.
Can we estimate coding sequence (CDS) density in a segment of DNA by measuring its (weighted) complexity as a sequence? (coding sequences constitute about 2% of the 300,000,000 long sequence of A,T,G,C which represents human genome)

Nucleotide triplets are distributed differently in regions with low/high frequency of coding sequences. (e.g. long runs of AAAAAA . . . are associated with intergenic regions of the genome).

Can we use these differences to detect/predict coding sequences when viewing the genome simply as a long string of data?
We introduce notion of topological pressure for finite sequences.

The topological pressure of a finite sequence is given by counting the number of distinct subwords at an exponentially shorter length, with weights determined by a locally constant function.
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Sequences with high topological pressure balance high complexity and high frequency of words which are weighted strongly.

Potential can be selected based on some underlying principle; e.g. GC content, or by training computationally against a data set. Potential determines a Markov measure as its equilibrium state, which we use to determine ‘coding potential’ (intron or exon).
We use a window of length around 66,000 bp on human genome, and obtain our parameters by optimizing the correlation between CDS density and topological pressure across these roughly 40,000 windows.
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Using these parameters, we compute the topological pressure along the genomes of fruit fly, monkey, etc... On Rhesus Macaque, correlation was 0.73:
### 2nd Base

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Which potential?