Open questions on Jacobians of curves over finite fields: $p$-ranks of curves

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Effective methods for abelian varieties
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Open questions on $p$-ranks of curves

Let $p$ be a prime number. Let $g$ be a natural number. Let $X$ be a curve defined over a finite field of characteristic $p$.

Open question B1:

If $X$ is a generic curve of genus $g$ and $p$-rank 0, what is the Newton polygon of $X$?

Open question B2:

What are the $p$-ranks of curves $X$ which are a cyclic $\mathbb{Z}/\ell$ cover of the projective line?

Outline. Definition of $p$-rank;
B0: how to compute $p$-rank with Cartier operator;
a generic Newton polygon,
$p$-ranks of cyclic covers of the projective line
The $p$-rank measures the number of $p$-torsion points on the Jacobian or the number of roots of the $L$-polynomial with $p$-adic absolute value 1.

**Fact/Def:** Let $X$ be a smooth $k$-curve of genus $g$.

Then $|J_X[p](k)| = p^f$ for some integer $0 \leq f \leq g$ called the $p$-rank of $X$.

Also, $f = \dim_{F_p} \text{Hom}(\mu_p, J_X[p])$ where $\mu_p \simeq \text{Spec}(k[x]/(x^p - 1))$ is the kernel of Frobenius on $\mathbb{G}_m$.

Let $L(t)$ be the $L$-polynomial of the zeta function of an $\mathbb{F}_q$-curve $X$.

The $p$-rank of $X$ is the length of the slope 0 portion of $\text{NP}(X)$.

$X$ is supersingular if all slopes of $\text{NP}(X)$ equal $1/2$.
$X$ supersingular implies $X$ has $p$-rank 0 but converse false for $g \geq 3$. 

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3 / 32
The moduli space $A_g$ of p.p. abelian varieties of dimension $g$ has dimension $\dim(A_g) = g(g + 1)/2$.

The $p$-rank 0 stratum of $A_g$ is irreducible of dimension $g(g - 1)/2$ (codimension $g$).

The supersingular locus of $A_g$ has dimension $\left\lfloor \frac{g^2}{4} \right\rfloor$ (number of components is a class number).

For $g \geq 3$, the dimension of the $p$-rank 0 strata is strictly bigger than the dimension of the supersingular strata.
Computing the $p$-rank

Let $C$ be the Cartier (semi-linear) operator on $H^0(X, \Omega^1)$.

Manin: the $p$-rank is $f = \dim(\text{Im}(C^g))$.
Thus: one can compute $f$, given $p$, $X$, and a basis of $H^0(X, \Omega^1)$.

Sage: compute the Cartier matrix, Hasse-Witt matrix, $p$-rank and $a$-number for hyperelliptic curve $X : y^2 = h(x)$ with $\deg(h(x)) = 2g + 1$.

```python
P. < x > = PolynomialRing(GF(67))
X = HyperellipticCurve(x^7 + x^3 + x)
X.p_rank()
2
```

The algorithm is a generalization of this fact for $g = 1$:
Let $h(x)$ be separable cubic polynomial.
$E : y^2 = h(x)$ has $p$-rank 0 iff the coeff of $x^{p-1}$ in $h(x)^{(p-1)/2}$ is 0.
Computing the $p$-rank of hyperelliptic curves

Let $p$ odd and $h(x) \in k[x]$ degree $2g + 1$ with no repeated roots.

Hyp. curve $X : y^2 = h(x)$: basis for $H^0(X, \Omega^1)$ is \{\( \frac{dx}{y}, \frac{xdx}{y}, \ldots, \frac{x^{g-1}dx}{y} \)\}.

Let $c_r$ be the coefficient of $x^r$ in the expansion of $h(x)^{(p-1)/2}$.

For $1 \leq t \leq g$, consider the $g \times g$ matrix $M_t$ s.t. $M_t(i,j) = c_p^{t-i}c_{p-i-j}$.

Yui:

The action of the Cartier operator on $H^0(X, \Omega^1)$ wrt this basis is $M_1$. $X$ is ordinary ($f = g$) if and only if $\det(M_1) \neq 0$.

The $p$-rank of $X$ is $f = \text{rank}(M)$ where $M = M_gM_{g-1} \cdots M_2M_1$.

B0: Need to fix! Achter/Howe found pervasive typo in literature (Yui, Zarhin, ...) Sage computes $M = M_1M_2 \cdots M_g$.
Example - Hermitian curve $X : y^q + y = x^{q+1}, \ q = p^n$

The Cartier operator $C$ acts on $H^0(X_q, \Omega^1)$.

Let $\Delta = \{(i,j) \mid i, j \in \mathbb{Z}, \ i, j \geq 0, \ i + j \leq q - 2\}$.

A basis for $H^0(X_q, \Omega^1)$ is $B = \{\omega_{i,j} := x^i y^j dx \mid (i,j) \in \Delta\}$.

Write $i = i_0 + pi^n_T$ and $j = j_0 + pj^n_T$ with $0 \leq i_0, j_0 \leq p - 1$.

$$C(x^i y^j dx) = x^{i_0} y^{j_0} C\left(x^{i_0}(x^{q+1} - y^q)j_0 dx\right)$$
$$= x^{i_0} y^{j_0} \sum_{l=0}^{j_0} \binom{j_0}{l} (-1)^l x^{p^n-1(j_0-l)} y^{p^{n-1}l} C\left(x^{i_0+j_0-l} dx\right).$$

$C(x^k dx) \neq 0$ iff $k \equiv -1 \mod p$. Need $i_0 + j_0 - l \equiv -1 \mod p$.

If $i_0 + j_0 < p - 1$, then $C(\omega_{i,j}) = 0$.

If $i_0 + j_0 \geq p - 1$, then $C(\omega_{i,j}) = \omega_{p^n-1(p-1-i_0) + i^n_T, p^n-1(i_0+j_0-(p-1)) + j^n_T}$. 

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Existence of curves with given genus and $p$-rank

The algorithm can be used to compute the $p$-rank of a fixed curve, but it is too complicated to be algebraically constructive. Let $g \in \mathbb{N}$, $0 \leq f \leq g$ and $p$ prime.

Let $\mathcal{M}_g^f$ (resp. $\mathcal{H}_g^f$) denote the $p$-rank $f$ strata of the moduli space of (hyperelliptic) curves of genus $g$.

**Theorem: Faber/Van der Geer**

Every component of $\mathcal{M}_g^f$ has dimension $2g - 3 + f$.

**Theorem: Glass/P ($p$ odd), P/Zhu ($p$ even)**

Every component of $\mathcal{H}_g^f$ has dimension $g - 1 + f$.

There exists a smooth (hyp.) curve over $\overline{\mathbb{F}}_p$ with genus $g$ and $p$-rank $f$.

In most cases, it is not known whether $\mathcal{M}_g^f$ and $\mathcal{H}_g^f$ are irreducible.
Let $A$ be the generic p.p. abelian variety of dimension $g$ and $p$-rank 0.

TFAE and true for $A$:

* the Newton polygon is $G_{1,g-1} \oplus G_{g-1,1}$ (slopes $\frac{1}{g}$ and $\frac{g-1}{g}$);
* the rank of the Cartier operator on $H^0(A, \Omega^1)$ is $g - 1$;
* the $a$-number of $A$ is 1.

B1 Open problem. For all $p$ and $g$,

1) are conditions * true for a generic curve of genus $g$ and $p$-rank 0?
2) does there exist a curve of genus $g$ and $p$-rank 0 satisfying *?

1) Yes: $g = 1, 2, 3$ for all $p$. (2) Yes: $g = 1, 2, 3, 4$ for all $p$. 
Existence of slopes 1/4 and 3/4

For all \( p \), there exists a smooth curve of genus 4 defined over \( \overline{\mathbb{F}_p} \) whose NP has slopes 1/4 and 3/4.

Let \( \mathcal{W} \) be moduli space of p.p. abelian 4-folds with action by \( \mathbb{Z}[\zeta_3] \) of signature \((3,1)\). Then \( \dim(\mathcal{W}) = 3 \).

Also \( \mathcal{W} \) is irreducible since \( \mathbb{Z}[\zeta_3] \) has class number 1.

Let \( S \) be moduli space of curves \( C_f : y^3 = f(x) \) (square-free \( f(x) \) degree 6). Then \( \dim(S) = 3 \).

The image of Torelli morphism on \( S \) is open, dense subspace of \( \mathcal{W} \).

There exists a point of \( \mathcal{W} \) representing abelian variety with slopes 1/4 and 3/4. Mantovan 2004 if \( p \) splits in \( \mathbb{Q}(\zeta_3) \), Bültel/Wedhorn 06 if \( p \) inert in \( \mathbb{Q}(\zeta_3) \), SAGE if \( p = 3 \).
Proof: inductive strategy, reduce to $p$-rank $f = 0$

Let $\nu_r$ be a NP type with $p$-rank 0 occurring in dimension $r$.

Let $c_r = \text{codim}(\mathcal{A}_g[\nu_r], \mathcal{A}_g)$.

For $g \geq r$, let $\nu_g$ be the NP type with $p$-rank $g - r$ ‘containing’ $\nu_r$

$(\nu_g = (G_{0,1} \oplus G_{1,0})^{g-r} \oplus \nu_r)$, add $g - r$ slopes of 0, 1.

Proposition P

If there exists a component $S_r$ of $\mathcal{M}_r[\nu_r]$ s.t. $\text{codim}(S_r, \mathcal{M}_r) = c_r$, then, for all $g \geq r$, there exists a component $S_g$ of $\mathcal{M}_g[\nu_g]$ s.t. $\text{codim}(S_g, \mathcal{M}_g) = c_r$. 
Newton polygon results for $f = g - 3$ and $f = g - 4$

Recall $v_{g,f} = f(G_{0,1} + G_{1,0}) + (G_{1,g-f-1} + G_{g-f-1,1})$.

Application - Achter/P. Let $g \geq 3$ and $f = g - 3$.

The generic point of any component of $\mathcal{M}^{g-3}_g$ has Newton polygon $v_{g,g-3}$ (slopes $0, \frac{1}{3}, \frac{2}{3}, 1$).

Application - Achter/P. Let $g \geq 4$ and $f = g - 4$.

The generic point of at least one component of $\mathcal{M}^{g}_f$ has Newton polygon $v_{g,g-4}$ (slopes $0, \frac{1}{4}, \frac{3}{4}, 1$).

Note: When $g = 4$, there is at most one component of $\mathcal{M}^{0}_4$ whose generic NP is not $v_{4,0}$. If so, the NP has slopes $\frac{1}{3}, \frac{1}{2}, \frac{2}{3}$. 

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A generic Newton polygon

If $f = g$, the NP is $g(G_{0,1} + G_{1,0})$ (slopes 0, 1).
If $f = g - 1$, the NP is $(g - 1)(G_{0,1} + G_{1,0}) + G_{1,1}$ (slopes $0, \frac{1}{2}, 1$).
If $f = g - 2$, the NP is $(g - 2)(G_{0,1} + G_{1,0}) + 2G_{1,1}$ (slopes $0, \frac{1}{2}, 1$).

If $0 \leq f \leq g - 3$, let $\nu_{g,f} = f(G_{0,1} + G_{1,0}) + (G_{1,g-f-1} + G_{g-f-1,1})$, (slopes 0, 1 with mult. $f$ and $\frac{1}{g-f}, \frac{g-f-1}{g-f}$ with mult. $g - f$.
Note that $\nu_{g,f}$ is the most generic Newton polygon with $p$-rank $f$.

$g = 4, f = 1.$

Conjecture: let $g \geq 3$ and $0 \leq f \leq g - 3$

The generic point of any component of $\mathcal{M}_g^f$ has Newton polygon $\nu_{g,f}$. 
Problem: Let $p$ odd and $X$ a hyp. curve of genus $g = 3$.

If $X$ generic $p$-rank 0, does Cartier matrix on $H^0(X, \Omega^1)$ have rank 2?

$X : y^2 = h(x)$ with $h(x) = x^7 + ax^6 + bx^5 + cx^4 + dx^3 + ex^2 + x$.
Finite-to-1 map $\mathbb{A}^5_k - \Delta \to \mathcal{H}_3$ taking $(a, b, c, d, e)$ to $X$.

The condition $M_2 M_1 M_0 = [0]$ true for dim 2 subspace in $(a, b, c, d, e)$.
For each component, does $M_0$ generically have rank 2?

Elkin/P: yes when $p = 3, 5$. If $p = 3$, Cartier operator has matrix

$$M_1 = \begin{bmatrix} e & 1 & 0 \\ b & c & d \\ 0 & 1 & a \end{bmatrix}.$$

If $r \leq 1$, then $e = b = d = a = 0$, and $X$ singular. So $r = 2$ if $f = 0$. 
Question: Let $\ell \neq p$ odd prime.

Does there exist a $\mathbb{Z}/\ell$-cover $Y \to \mathbb{P}^1$ over $\overline{\mathbb{F}}_p$ such that $Y$ is a smooth curve of genus $g$ and $p$-rank $f$?

Not always! There are new constraints on $g$ and $f$.

Equation: $y^\ell = \prod_{i=1}^{n} (x - \beta_i)^{a_i}$ where $0 < a_i < \ell$ and $\sum_{i=1}^{n} a_i \equiv 0 \mod \ell$.

$a_1, \ldots, a_n$ well-defined up to permutation and sim. mult. by $c \in (\mathbb{Z}/\ell\mathbb{Z})^*$. 

Def: Inertia type: $\vec{a} = \{a_1, \ldots, a_n\}$.

Riemann-Hurwitz: $g(Y) = (\ell - 1)(n - 2)/2$.

Congruence condition

Let $e$ be the order of $p$ modulo $\ell$. Then $e$ divides the $p$-rank $f$. 

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Eigenspaces

Let $\tau \in \text{Aut}(Y)$ with $\tau(y) = \zeta y$ where $\zeta$ is primitive $\ell$th root of unity.

Then $\tau$ induces a linear transformation of $H^0(Y, \Omega^1)$.

Decompose $H^0(Y, \Omega^1)$ into eigenspaces: $H^0(Y, \Omega^1) = \bigoplus_{i=0}^{\ell-1} L_i$, where $L_i = \{ \omega \in H^0(Y, \Omega^1) \mid \tau^*(\omega) = \zeta^i \omega \}$. 

**Fact:** Let $d_i = \dim(L_i)$. Then $d_0 = 0$ and $d_i = -1 + \sum_{j=1}^{n} \left( \frac{ia_j}{\ell} - \left\lfloor \frac{ia_j}{\ell} \right\rfloor \right)$. 

**Ex:** Let $\ell = 3$.

The *signature type* is $(r, s)$ where $r = \dim(L_1)$ and $s = \dim(L_1)$. 

Note $r + s = g$ and $(g - 1)/3 \leq r, s \leq (2g + 1)/3$. 

There is a bijection between inertia types and signature types: 

$$\#\{a_i = 1\} = 2s - r + 1, \quad \#\{a_i = 2\} = 2r - s + 1.$$
Upper bound on $p$-rank

Cartier operator $C$ permutes $\{L_i \mid 1 \leq i \leq \ell - 1\}$ by $C(\zeta^i \omega) = \zeta^i C(\omega)$ so $C(L_i) \subset L_{\sigma(i)}$ where $\sigma$ is the permutation $i \mapsto p^{-1} i \mod \ell$ of $(\mathbb{Z}/\ell\mathbb{Z})^*$. Each orbit of $\{L_i \mid 1 \leq i \leq \ell - 1\}$ under $C$ has length $e$, where $e$ is the order of $p$ modulo $\ell$.

**Bouw: (dual result to action of $F$ on $H^1(Y, O)$)**

The stable rank of $C$ on $L_i$ is bounded by $\min\{\dim(L_i)\}$ across orbit.

Let $B(\vec{a}) = \sum_{\text{orbits } O} e \cdot \min\{\dim(L_i) \mid i \in O\}$.

Then $f(Y) \leq B(\vec{a})$ for all $\mathbb{Z}/\ell$-covers with inertia type $\vec{a}$.

The upper bound $B(\vec{a})$ occurs as the $p$-rank for (generic) curve in $\mathcal{I}_{\ell,\vec{a}}$.

**Ex:** Let $\ell = 3$.

Then $B(\vec{a}) = g$ if $p \equiv 1 \mod 3$ and $B(\vec{a}) = 2\min\{r, s\}$ if $p \equiv 2 \mod 3$. 

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An existence result for trielliptic covers

Let \( g \geq 3 \) and let \((r, s)\) be a trielliptic signature for \( g \).

Suppose either:

1. \( p \equiv 2 \mod 3 \) is odd and \( 0 \leq f \leq 2\min(r, s) \) is even; or
2. \( p \equiv 1 \mod 3 \) and \( f = g - 2 \).

Ozman/P/Weir

Then there exists a \( \mathbb{Z}/3 \)-cover \( \phi : Y \to \mathbb{P}^1 \) with \( Y \) a smooth curve of genus \( g \), trielliptic signature \((r, s)\) and \( p \)-rank \( f \).

More generally, \( \mathcal{T}^f_{(r, s)} \) is non-empty and contains a component \( S \) with \( \dim(S) = \max(r, s) - 1 + f/2 \) in case (1) and \( \dim(S) = f \) in case (2).
Let \( \ell \neq p \) be prime.

Restrict to \( \mathbb{Z}/\ell \)-covers of the projective line with 3 branch points. (There are only finitely many of these).

**Open question**

For \( \vec{a} = (a_1, a_2, a_3) \) inertia type for \( \mathbb{Z}/\ell \), what is the \( p \)-rank of the \( \mathbb{Z}/\ell \)-cover \( Y \to \mathbb{P}^1 \) with inertia \( \vec{a} \)?

Can suppose that \( a_2 = 1 \).
Reduce to equation \( y^\ell = x^{a_1}(x - 1)^1 = x^{a_1+1} - x^{a_1} \).

Elkin: formula for Cartier operator on \( H^0(X, \Omega^1) \).
Example: trielliptic $g = 4$, signature $(2,2)$

If $g = 4$ and $\dim(L_1) = \dim(L_2) = 2$:

(Note - Torelli locus has codimension 1 in $GU(2,2)$).

Write $X : y^3 = p_1(x)p_2(x)^2$ where $p_1(x) = x(x^2 - 1)$

and $p_2(x) = x^3 + ax^2 + bx + c$ has distinct roots in $k - \{0, \pm 1\}$.

Basis $\{w_{11} = \frac{dx}{y}, w_{12} = \frac{x dx}{y}\}$ and $\{w_{21} = p_1(x)\frac{dx}{y^2}, w_{22} = p_1(x)\frac{x dx}{y^2}\}$.

Elkin: action of $C$ on basis.

$$C(w_{11}) = f_{1,p-1}(x)w_{21}, \quad C(w_{12}) = f_{1,p-2}(x)w_{21},$$

$$C(w_{21}) = f_{2,p-1}(x)w_{11}, \quad C(w_{22}) = f_{2,p-2}(x)w_{11}.$$

The $p$-rank of $X$ is rank of matrix $M = C^{(p^3)}C^{(p^2)}C^{(p)}C$. 

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Example: trielliptic $g = 4$ signature $(2, 2)$, $p = 5$

Curve $X : y^3 = x(x^2 - 1)p_2(x)^2$ where $p_2(x) = x^3 + ax^2 + bx + c$.
When $p = 5$, the Cartier matrix $C$ is

$$
\begin{bmatrix}
0 & 0 & 4b & 2a + c \\
0 & 0 & 4c & 2b + 3 \\
4abc + 4b^3 + 3bc^2 + 2c^2 & a^3 + ab + 2a + 3c & 0 & 0 \\
2ac^2 + 2b^2c + c^3 & 3a^2b + 2a^2 + ac + 3b^2 + 2b & 0 & 0
\end{bmatrix}.
$$

$$
\text{det}(C) = (ab + 2c^2 + 2)^2 \text{disc}(p_2(x)).
$$

Problem: $M = C^{(125)} C^{(25)} C^{(5)} C$ has 8 non-trivial entries, each a polynomial in 3 variables with 288 monomials, half of degree 208, the other half of degree 416.

Strategy: use resultants, lift solutions.
Example: trielliptic $g = 4$ signature $(2,2)$, $p = 5$

Ozman/P/Weir: Let $p = 5$.

The $p$-rank 0 strata of the moduli space $T_{2,2}$ of genus 4 trielliptic curves with signature $(2,2)$ has two components, each rational of dimension 1, each intersecting $\Delta_2$.

Proof: Show $V = \{(a, b, c) \in \mathbb{A}^3 : M = 0\}$ has 4 components.

Action of $S_3 = \text{Stab}(0, 1, -1)$ permutes 3 of them and fixes one.

The two irreducible components $Z$ and $W$ of $T_{2,2}$ parametrized by $X : y^3 = x(x^2 - 1)(Dx^3 + Ax^2 + Bx + C)^2$ are

\[
\begin{aligned}
A &= 2u^{10}v + 2u^6v^5 + v^{11} \\
B &= u^5v^6 \\
C &= u^6v^5 \\
D &= 3u^{11} + u^5v^6 + 4uv^{10}
\end{aligned}
\quad\quad
\begin{aligned}
A &= u^2v + 4v^3 \\
B &= uv^2 \\
C &= u^2v \\
D &= u^3 + 2uv^2
\end{aligned}
\]
Open question B1: $X$ a generic curve of genus $g$ and $p$-rank 0
what is the Newton polygon of $X$?

Guess: NP has slopes $1/g$ and $(g - 1)/g$.
equivalently, guess rank of Cartier operator on $H^0(X, \Omega^1)$ is $g - 1$.

Test case: generic hyperelliptic curve with $g = 3$ and $f = 0$.

Open question B2: for $\mathbb{Z}/\ell$-covers $X \to \mathbb{P}^1$
What is $p$-rank of $X$?

New conditions on $g$ and $f$ for given $\ell$, $p$.

Big picture: in moduli space $\mathcal{A}_g$, study interaction between Torelli locus, $p$-rank strata, Hurwitz spaces, and loci of abelian varieties which decompose (with product polarization).
Geometric tools: dimension of \( p \)-rank strata

Let \( p \) prime and let \( \ell \neq p \) be odd prime.
Let \( e \) be the order of \( p \) modulo \( \ell \).
Let \( g \) be multiple of \((\ell - 1)/2\) and \( 0 \leq f \leq g \) multiple of \( e \).

Let \( T_{\ell, \vec{a}} \) be the Hurwitz space of \( \mathbb{Z}/\ell \)-covers of \( \mathbb{P}^1_k \) with inertia type \( \vec{a} \).
Let \( \Gamma \) be a component of the \( p \)-rank \( f \) strata \( T^f_{\ell, \vec{a}} \).

Oort Purity: The Newton polygon can change only in codim 1.

\[
\dim(\Gamma) \geq \dim(T_{\ell, \vec{a}}) - \left( B(\vec{a}) - f \right)/e.
\]

**Ex:** Let \( \ell = 3 \) and \( f \) even and \( p \equiv 2 \) mod 3. Then

\[
\dim(\Gamma) \geq g - 1 - \min(r, s) + f/2.
\]

Strategy: prove existence of cyclic covers with given genus and \( p \)-rank by proving equality for dimension.
Suppose $X$ has two components $X_1$ and $X_2$ (of genera $g_1$ and $g_2$ and $p$-ranks $f_1$ and $f_2$)

which intersect in exactly one ordinary double point $P$ (a ramification point under $\mathbb{Z}/\ell$-action on $X_1$ and on $X_2$).

[BLR] $J_X \simeq J_{X_1} \times J_{X_2}$.

So $g = g_1 + g_2$ and $f = f_1 + f_2$.

Let $\Delta_{g_1}$ be image of clutching morphism:

$k_{g_1,g_2} : \tilde{T}_{\ell,g_1} \times \tilde{T}_{\ell,g_2} \to \tilde{T}_{\ell,g_1+g_2}$ where $\Downarrow \times \Downarrow \mapsto \Rightarrow$.

For $\ell \geq 3$, have admissible condition for inertia types at node.
Suppose $X$ has two components $X_1$ and $X_2$ (of genera $g_1$ and $g_2$ and $p$-ranks $f_1$ and $f_2$) which intersect in $\ell$ ordinary double points $P$ and $Q$ (an orbit under $\mathbb{Z}/\ell$-action on $X_1$ and on $X_2$).

[BLR] $1 \to (\mathbb{G}_m)^{\ell-1} \to J_X \to J_{X_1} \times J_{X_2} \to 1$.

So $g = g_1 + g_2 + (\ell - 1)$ and $f = f_1 + f_2 + (\ell - 1)$.

Let $\Xi_{g_1}$ be image of clutching morphism:

$\lambda_{g_1,g_2} : \overline{T}_{\ell,g_1;1} \times \overline{T}_{\ell,g_2;1} \to \overline{T}_{\ell,g_1+g_2+(\ell-1)}$ where $\langle \rangle \times \circlearrowleft \to \bigcirc$. 
Suppose \( X \) is a singular curve of genus \( g \).

Then \( X \) could be reducible.

\([\Delta_i]\) For example, it could consist of two irreducible components \( X_1 \) of genus \( i \) and \( X_2 \) of genus \( g - i \) intersecting in exactly one ordinary double point.

Or \( X \) could be irreducible.

\([\Delta_0]\) For example, it could self-intersect in an ordinary double point with normalization an irreducible curve of genus \( g - 1 \).
Suppose $X$ has two components $X_1$ and $X_2$ (of genera $g_1$ and $g_2$) which intersect in exactly one ordinary double point $P$. Then $J_X \simeq J_{X_1} \times J_{X_2}$. So $g = g_1 + g_2$.

Let $\Delta_{g_1}$ be image of clutching morphism:

$$\kappa_{g_1,g_2} : \overline{M}_{g_1;1} \times \overline{M}_{g_2;1} \to \overline{M}_{g_1+g_2}$$

where $\to \to \to \to \to \to \to \to \to$.
Suppose $X$ self-intersects in one ordinary double point $P$.

Its normalization $X_1$ is a curve (of genus $g_1$).

Then $1 \to \mathbb{G}_m \to J_X \to J_{X_1} \to 1$.

So $g = g_1 + 1$ and $f = f_1 + 1$.

Let $\Xi_0$ be image of clutching morphism:

$\kappa_g : \overline{M}_{g;2} \to \overline{M}_{g+1}$ where $\leftarrow \leftrightarrow \Rightarrow$. 
Geometry of boundary

\[ \kappa_{g_1,g_2} : \mathcal{M}_{g_1;1} \times \mathcal{M}_{g_2;1} \to \Delta_{g_1}[\mathcal{M}_{g_1} + g_2] \]

\[ \kappa_{g} : \mathcal{M}_{g-1;2} \to \Delta_0[\mathcal{M}_g] \]

Then \( \Delta_i \) is an irreducible divisor in \( \mathcal{M}_g \).

Let \( \partial \mathcal{M}_g = \bigcup_{i=0}^{g/2} \Delta_i \) and \( \mathcal{M}_g^0 = \mathcal{M}_g - \partial \mathcal{M}_g \).

Then \( \mathcal{M}_g^0 \) is the moduli space of smooth curves of genus \( g \).
Let $S$ be a component of $\mathcal{M}_g^f$.
We prove that $S$ intersects $\partial \mathcal{M}_g$ in every way possible.
Also have similar result about boundary of $\mathcal{H}_g^f$ when $p > 2$.

**Theorem (Achter/P)**

Let $g_i \in \mathbb{Z}^{\geq 1}$ and $0 \leq f_i \leq g_i$ be such that $\sum g_i = g$ and $\sum f_i = f$. Then $S$ contains a chain of smooth curves $Y_i$ of genus $g_i$ and $p$-rank $f_i$.

**Sketch of proof:**
When $f = 0$, follows from result of Faber/van der Geer.

When $f \geq 1$, then $\dim S > 2g - 3$.
So $S$ intersects $\Delta_0$, again by F/vdG.

$$1 \rightarrow \mathbb{G}_m \rightarrow J(\mathcal{H}) \rightarrow J(\mathcal{G}) \rightarrow 1$$
If $f \geq 1$, inductive strategy:

$M^f_g$
If $f \geq 1$, inductive strategy:

$M_f^g$ degenerate
Sketch of proof - continued

If $f \geq 1$, inductive strategy:

\[ M^{f-1}_{g-1;2} \rightarrow M^f_g \]

degenerate

unclutch
If $f \geq 1$, inductive strategy:

$M_{g-1;2}^{f-1}$

$M_g^f$

degenerate

unclutch

degenerate

key step!
If $f \geq 1$, inductive strategy:

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If $f \geq 1$, inductive strategy:

\[ \mathcal{M}_{g-1;2}^f \xrightarrow{\text{degenerate}} \mathcal{M}_g^f \]

unclutch

degenerate

move

clutch

key step!