Sparse Approximation of PDEs based on Compressed Sensing

Simone Brugiapaglia
Department of Mathematics
Simon Fraser University

Retreat for Young Researchers in Stochastics
September 24, 2016
Introduction

We address the following question

Can we employ Compressed Sensing to solve a PDE?

In particular, we consider the weak formulation of a PDE

\[
\text{find } u \in U : \quad a(u, v) = F(v), \quad \forall v \in V,
\]

focusing on the Petrov-Galerkin (PG) discretization method [Aziz and Babuška, 1972].

Motivation:

► reduce the computational cost associated with a classical PG discretization;

► situations with a limited budget of evaluations of $F(\cdot)$;

► better theoretical understanding of the PG method.

Case study:

Advection-diffusion-reaction (ADR) equation, with

$U = V = H^1_0(\Omega), \quad \Omega = [0, 1]^d$, and

\[
a(u, v) = (\eta \nabla u, \nabla v) + (b \cdot \nabla u, v) + (\rho u, v), \quad F(v) = (f, v).
\]
Compressed Sensing

CORSING (COmpRessed SolvING)

A theoretical study of CORSING
Compressed Sensing (CS)

[D. Donoho, 2006; E. Candès, J. Romberg, and T. Tao, 2006]

Consider a signal $s \in \mathbb{C}^N$, sparse w.r.t. $\Psi \in \mathbb{C}^{N \times N}$:

$$s = \Psi u \quad \text{and} \quad \|u\|_0 =: s \ll N,$$

where $\|u\|_0 := \#\{i : u_i \neq 0\}$.

It can be acquired by means of $m \ll N$ linear and non-adaptive measurements

$$\langle s, \varphi_i \rangle =: f_i, \quad \text{for } i = 1, \ldots, m.$$

If we consider the matrix $\Phi = [\varphi_i] \in \mathbb{C}^{N \times m}$, we have

$$Au = f,$$

where $A = \Phi^H \Psi \in \mathbb{C}^{m \times N}$ and $f \in \mathbb{C}^m$. 

A sparse vector $u$

\begin{center}
\begin{tikzpicture}
    \draw[->] (0,0) -- (10,0) node[right] {component $i$};
    \draw[->] (0,-1) -- (0,2) node[above] {$z$};
    \foreach \x in {0,20,40,60,80,100}
        \draw[fill=white] (\x,-0.1) -- (\x,0.1) circle (2pt);
    \foreach \y in {0,0.5,1,1.5,2}
        \draw[fill=white] (-0.1,\y) -- (0.1,\y) circle (2pt);
\end{tikzpicture}
\end{center}
Sensing phase

Since $m \ll N$, the system $A\mathbf{u} = \mathbf{f}$ is highly underdetermined. How to recover the right $\mathbf{u}$ among its infinite solutions?
Recovery: finding a needle in a haystack

Thanks to the sparsity hypothesis, we can resort to **sparse recovery techniques**. We aim at approximating the solution to

\[
(P_0) \quad \min_{u \in \mathbb{C}^N} \|u\|_0, \quad \text{s.t. } Au = f.
\]

⚠️ Unfortunately, in general \((P_0)\) is a **NP-hard** problem...

😊 There are **computationally tractable** strategies to approximate it!

In particular, we employ the **greedy algorithm Orthogonal Matching Pursuit (OMP)** to approximate

\[
(P_0^\varepsilon) \quad \min_{u \in \mathbb{C}^N} \|u\|_0 \quad \text{or} \quad (P_0^s) \quad \min_{u \in \mathbb{C}^N} \|Au - f\|_2 \\
\text{s.t. } \|Au - f\|_2 \leq \varepsilon \quad \text{s.t. } \|u\|_0 \leq s.
\]

Another valuable option is **convex relaxation** (not discussed here)

\[
(P_1) \quad \min_{u \in \mathbb{C}^N} \|u\|_1, \quad \text{s.t. } Au = f.
\]
Orthogonal Matching Pursuit (OMP)

Input:
Matrix \( A \in \mathbb{C}^{m \times N} \), with \( \ell^2 \)-normalized columns
Vector \( f \in \mathbb{C}^m \)
Tolerance on the residual \( \epsilon > 0 \) (or else, sparsity \( s \in [N] \))

Output:
Approximate solution \( u \) to \((P_0^\epsilon)\) (or else, \((P_0^s)\))

Procedure:
1: \( S \leftarrow \emptyset \) \hspace{1cm} \triangleright \text{Initialization}
2: \( u \leftarrow 0 \)
3: \textbf{while } \| Au - f \|_2 > \epsilon \text{ (or else, } \| u \|_0 < s \) \textbf{ do}
4: \( \tilde{j} \leftarrow \arg \max_{j \in [N]} |[A^H(Au - f)]_j| \) \hspace{1cm} \triangleright \text{Select new index}
5: \( S \leftarrow S \cup \{ \tilde{j} \} \) \hspace{1cm} \triangleright \text{Enlarge support}
6: \( u \leftarrow \arg \min_{z \in \mathbb{C}^N} \| Az - f \|_2 \text{ s.t. } \text{supp}(z) \subseteq S \) \hspace{1cm} \triangleright \text{Minimize residual}
7: \textbf{end while}
8: \textbf{return } u

\triangleright \text{The computational cost for the } (P_0^s) \text{ formulation is in general } \mathcal{O}(smN).
Recovery results based on the RIP

Many important recovery results in CS are based on the **Restricted Isometry Property (RIP)**.

**Definition (RIP)**

A matrix $A \in \mathbb{C}^{m \times N}$ satisfies the RIP $(s, \delta)$ iff

$$(1 - \delta)\|u\|_2^2 \leq \|Au\|_2^2 \leq (1 + \delta)\|u\|_2^2, \quad \forall u \in \Sigma_s^N := \{v \in \mathbb{C}^N : \|v\|_0 \leq s\}.$$ 

Among many others, the RIP implies the following recovery result for OMP. [T. Zhang, 2011; A. Cohen, W. Dahmen, R. DeVore, 2015]

**Theorem (RIP $\Rightarrow$ OMP recovery)**

*There exist $K \in \mathbb{N}$, $C > 0$ and $\delta \in (0, 1)$ s.t. for every $s \in \mathbb{N}$, the following holds: if*

$$A \in RIP((K + 1)s, \delta),$$

*then, for any $f \in \mathbb{C}^m$, the OMP algorithm computes in $Ks$ iterations a solution $u$ that fulfills*

$$\|Au - f\|_2 \leq C \inf_{w \in \Sigma_s^N} \|Aw - f\|_2.$$
Compressed Sensing

CORSING (COmpRessed SolvING)

A theoretical study of CORSING
The reference problem

Given two Hilbert spaces $U, V$, consider the following problem

$$\text{find } u \in U : a(u, v) = \mathcal{F}(v), \quad \forall v \in V, \quad (1)$$

where $a : U \times V \to \mathbb{R}$ is a bilinear form and $\mathcal{F} \in V^\ast$. We will assume $a(\cdot, \cdot)$ to fulfill

$$\exists \alpha > 0 : \inf_{u \in U} \sup_{v \in V} \frac{a(u, v)}{\|u\|_U \|v\|_V} \geq \alpha, \quad (2)$$

$$\exists \beta > 0 : \sup_{u \in U} \sup_{v \in V} \frac{|a(u, v)|}{\|u\|_U \|v\|_V} \leq \beta, \quad (3)$$

$$\sup_{u \in U} a(u, v) > 0, \quad \forall v \in V \setminus \{0\}. \quad (4)$$

\( (2) + (3) + (4) \implies \exists! \text{ solution to (1). [J. Nečas, 1962]} \)

We will focus on advection-diffusion-reaction (ADR) equations.
The Petrov-Galerkin method

Given $\Omega \subseteq \mathbb{R}^d$, consider the weak formulation of an ADR equation:

$$\text{find } u \in H_0^1(\Omega) : \left( \eta \nabla u, \nabla v \right) + \left( b \nabla u, v \right) + (\rho u, v) = (f, v), \forall v \in H_0^1(\Omega).$$

(ADR)

Choose $U^N \subseteq H_0^1(\Omega)$ and $V^M \subseteq H_0^1(\Omega)$ with

$$U^N = \text{span}\{\psi_1, \ldots, \psi_N\}, \quad V^M = \text{span}\{\varphi_1, \ldots, \varphi_M\}$$

Then we can discretize (ADR) as

$$A\widehat{u} = f, \quad A_{ij} = a(\psi_j, \varphi_i), \quad f_i = F(\varphi_i)$$

with $A \in \mathbb{C}^{M \times N}$, $f \in \mathbb{C}^M$.

A common choice is $M = N$.

- **Examples of Petrov-Galerkin methods:** Finite elements, spectral methods, collocation methods, etc.
The main analogy

A fundamental analogy guided us through the development of our method...

\begin{align*}
\text{Petrov-Galerkin method:} & \quad \text{Sampling:} \\
\text{solution of a PDE} & \quad \text{signal} \\
\text{tests (bilinear form)} & \quad \text{measurements (inner product)}
\end{align*}

Reference:

*Compressed solving: a numerical approximation technique for elliptic PDEs based on compressed sensing*

S. B., S. Micheletti, S. Perotto

Related literature

Ancestors: PDE solvers based on $\ell^1$-minimization

Inviscid Burgers’ equation, conservation laws

Hamilton-Jacobi, transport equation

CS techniques for PDEs

2010  [S. Jokar, V. Mehrmann, M. Pfetsch, and H. Yserentant, 2010]
Recursive mesh refinement based on CS (Poisson equation)

Application of CS to parametric PDEs and Uncertainty Quantification

CORSING for ADR problems
CORSING (COmpRessed SolvING)

Assembly phase

1. Choose two sets of $N$ independent elements of $U$ and $V$:
   \[ \text{trials} \rightarrow \{\psi_1, \ldots, \psi_N\}, \quad \{\varphi_1, \ldots, \varphi_N\} \leftarrow \text{tests}; \]

2. choose $m \ll N$ tests $\{\varphi_{\tau_1}, \ldots, \varphi_{\tau_m}\}$:

   \[ \begin{array}{c}
   \text{DETERMINISTICALLY} \\
   \downarrow \\
   \text{D-CORSING} \\
   \end{array} \quad \begin{array}{c}
   \text{how?} \quad \Rightarrow \quad \text{randomly} \\
   \downarrow \\
   \text{R-CORSING} \\
   \end{array} \]

3. build $A \in \mathbb{C}^{m \times N}$ and $f \in \mathbb{C}^m$ as
   \[ [A]_{ij} := a(\psi_j, \varphi_{\tau_i}) \quad [f]_i := F(\varphi_{\tau_i}). \]

Recovery phase

Find a compressed solution $u^N_m$ to $Au = f$, via sparse recovery.
CORSING (COmpRessed SolvING)

Assembly phase

1. Choose two sets of $N$ independent elements of $U$ and $V$:
   \[
   \text{trials} \rightarrow \{\psi_1, \ldots, \psi_N\}, \quad \{\varphi_1, \ldots, \varphi_N\} \leftarrow \text{tests};
   \]

2. choose $m \ll N$ tests $\{\varphi_{\tau_1}, \ldots, \varphi_{\tau_m}\}$:
   \[
   \text{DETERMINISTICALLY}
   \]
   \[
   \text{D-CORSING}
   \]
   \[
   \text{randomly}
   \]
   \[
   \text{R-CORSING}
   \]

3. build $A \in \mathbb{C}^{m \times N}$ and $f \in \mathbb{C}^m$ as
   \[
   [A]_{ij} := a(\psi_j, \varphi_{\tau_i}) \quad [f]_i := F(\varphi_{\tau_i}).
   \]

Recovery phase

Find a compressed solution $u_m^N$ to $Au = f$, via sparse recovery.
Classical case: square matrices

When dealing with Petrov-Galerkin discretizations, one usually ends up with a **big square** matrix.

\[
\begin{bmatrix}
\psi_1 & \psi_2 & \psi_3 & \psi_4 & \psi_5 & \psi_6 & \psi_7 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
\varphi_1 \rightarrow & \times & \times & \times & \times & \times & \times & \times \\
\varphi_2 \rightarrow & \times & \times & \times & \times & \times & \times & \times \\
\varphi_3 \rightarrow & \times & \times & \times & \times & \times & \times & \times \\
\varphi_4 \rightarrow & \times & \times & \times & \times & \times & \times & \times \\
\varphi_5 \rightarrow & \times & \times & \times & \times & \times & \times & \times \\
\varphi_6 \rightarrow & \times & \times & \times & \times & \times & \times & \times \\
\varphi_7 \rightarrow & \times & \times & \times & \times & \times & \times & \times \\
\end{bmatrix}
\begin{bmatrix}
\psi_j, \varphi_i \\
\end{bmatrix} =
\begin{bmatrix}
u_1 \\
u_2 \\
u_3 \\
u_4 \\
u_5 \\
u_6 \\
u_7 \\
\end{bmatrix} =
\begin{bmatrix}
\mathcal{F}(\varphi_1) \\
\mathcal{F}(\varphi_2) \\
\mathcal{F}(\varphi_3) \\
\mathcal{F}(\varphi_4) \\
\mathcal{F}(\varphi_5) \\
\mathcal{F}(\varphi_6) \\
\mathcal{F}(\varphi_7) \\
\end{bmatrix}
“Compressing” the discretization

We would like to use only $m$ random tests instead of $N$, with $m \ll N$...

\[
\begin{array}{cccccccc}
\psi_1 & \psi_2 & \psi_3 & \psi_4 & \psi_5 & \psi_6 & \psi_7 \\
\Downarrow & \Downarrow & \Downarrow & \Downarrow & \Downarrow & \Downarrow & \Downarrow \\
\varphi_1 \rightarrow & \times & \times & \times & \times & \times & \times & \times \\
\varphi_2 \rightarrow & \times & \times & \times & \times & \times & \times & \times \\
\varphi_3 \rightarrow & \times & \times & \times & \times & \times & \times & \times \\
\varphi_4 \rightarrow & \times & \times & \times & \times & \times & \times & \times \\
\varphi_5 \rightarrow & \times & \times & \times & \times & \times & \times & \times \\
\varphi_6 \rightarrow & \times & \times & \times & \times & \times & \times & \times \\
\varphi_7 \rightarrow & \times & \times & \times & \times & \times & \times & \times \\
\end{array}
\]

\[
a(\psi_j, \varphi_i) = \begin{bmatrix}
    u_1 \\
u_2 \\
u_3 \\
u_4 \\
u_5 \\
u_6 \\
u_7
\end{bmatrix} = \begin{bmatrix}
F(\varphi_1) \\
F(\varphi_2) \\
F(\varphi_3) \\
F(\varphi_4) \\
F(\varphi_5) \\
F(\varphi_6) \\
F(\varphi_7)
\end{bmatrix}
\]
Sparse recovery

...in order to obtain a reduced discretization.

\[
\begin{align*}
\psi_1 & \psi_2 & \psi_3 & \psi_4 & \psi_5 & \psi_6 & \psi_7 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
\varphi_2 & \rightarrow & \begin{bmatrix}
\times & \times & \times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times & \times & \times
\end{bmatrix} \\
\varphi_5 & \rightarrow & \underbrace{a(\psi_j,\varphi_i)} & \begin{bmatrix}
u_1 \\
u_2 \\
u_3 \\
u_4 \\
u_5 \\
u_6 \\
u_7
\end{bmatrix} = \begin{bmatrix}
F(\varphi_2) \\
F(\varphi_5)
\end{bmatrix}
\end{align*}
\]

The solution is then computed using sparse recovery techniques.
How to choose \( \{ \psi_j \} \) and \( \{ \varphi_i \} \)?
How to choose \( \{ \psi_j \} \) and \( \{ \varphi_i \} \)?

One heuristic criterion commonly used in CS is to choose one basis sparse in space, and the other in frequency.

Hierarchical hat functions
[O. Zienkiewicz et al., 1982]

We name the corresponding strategies CORSING \( \mathcal{HS} \) and \( S\mathcal{H} \).
A 1D example

We test CORSING $\mathcal{HS}$ on the homogeneous 1D Poisson problem $(a(u, v) = (u', v'))$:

- Trial space dimension $N = 8191$
- Solution sparsity $s = 50$
- Selected random tests $m = 1200$

**Test Savings:** $TS := \frac{N - m}{N} \cdot 100\% \approx 85\%$

\[ \times = \text{hat functions selected by OMP after solving the program} \]

\[ \min \| Au - f \|_2, \quad \text{s.t.} \| u \|_0 \leq 50 \]
A glance at the space of coefficients...

Lexicographic ordering

![Lexicographic ordering graph]

Level-based ordering \((\log_{10}|\hat{u}_{\ell,k}|)\)

![Level-based ordering graphs]
Generalization to the 2D case (space domain)

Hierarchical Pyramids
[H. Yserentant, 1986]

Tensor product of hat functions

\(P\)

\(Q\)
The 2D case (frequency domain)

Tensor product of sine functions

We have four strategies: CORSING $\mathcal{P}S$, $QS$, $SP$ and $SQ$. 
An advection-dominated example

We evaluate the CORINGN performance on the following 2D advection-dominated problem

\[
\begin{cases}
-\mu \Delta u + b \cdot \nabla u = f & \text{in } \Omega = (0, 1)^2, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]

where \( b = [1, 1]^\top, \ 0 < \mu \ll 1 \) and \( f \) s.t. the exact solution be

\[
u_{\mu}^*(x) = C_\mu (x_1 - x_1^2)(x_2 - x_2^2)(e^{x_1/\mu} + e^{x_2/\mu} - 2),
\]

where \( C_\mu > 0 \) is chosen such that \( \max_{x \in \Omega} u_{\mu}^*(x) = 1. \)

- The function \( u_{\mu}^* \) exhibits two boundary layers along the edges \( \{x_1 = 1\} \) and \( \{x_2 = 1\} \) of \( \Omega. \)
\[ N = 16129 \]
\[ TS = 85\% \]
\[ ESP = 1.00 \]
\[ L^2\text{-rel. err.} = 7.1e^{-02} \]

\[ N = 16129 \]
\[ TS = 90\% \]
\[ ESP = 0.94 \]
\[ L^2\text{-rel. err.} = 8.7e^{-02} \]

**Figure:** CORSING \( \mathcal{SP} \), with \( \mu = 0.01 \): worst solution in the successful cluster (right). 50 random experiments are performed.

\[ ESP = \text{Empirical Success Probability} \]
Cost reduction with respect to the full-PG (m=N)

We compare the assembly/recovery times of full-PG and CORSING.

<table>
<thead>
<tr>
<th></th>
<th>full-PG</th>
<th>CORSING $SP$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>$f$</td>
<td>$t_{\text{rec}}$ (%)</td>
</tr>
<tr>
<td>2.5e+03</td>
<td>9.1e-01</td>
<td>7.1e+01</td>
</tr>
<tr>
<td>2.5e+02</td>
<td>2.0e-01</td>
<td>3.4e+01</td>
</tr>
</tbody>
</table>

- The assembly time reduction is proportional to $TS$.
- Also the RAM is reduced proportionally to $TS$.
- The recovery phase is cheaper for high $TS$ rates.

The CORSING method can considerably reduce the computational cost associated with a full-PG discretization.
More challenging test cases

The CORSING technique has also been implemented for

**The 3D Poisson problem**

\[
\begin{align*}
-\Delta u &= f \quad \text{in } \Omega = (0, 1)^3 \\
  u &= 0 \quad \text{on } \partial \Omega
\end{align*}
\]

![CORSING QS TS=85%](image1)

Exact solution

**The 2D Stokes problem**

\[
\begin{align*}
-\Delta u + \nabla p &= f \quad \text{in } \Omega = (0, 1)^2 \\
  \text{div} u &= 0 \quad \text{in } \Omega \\
  u &= 0 \quad \text{on } \partial \Omega
\end{align*}
\]

![CORSING SP TS=70%](image2)

Exact solution
Compressed Sensing

CORSING (COmpRessed SolvING)

A theoretical study of CORSING
A theoretical understanding of the method

Reference:
*A theoretical study of COmpRessed SolvING for advection-diffusion-reaction problems*
S.B., F. Nobile, S. Micheletti, S. Perotto
To appear in Mathematics of Computation

Some notation:

- Finite dimensional *trial* and *test* spaces
  
  $$U^N := \text{span}\{\psi_j\}_{j \in [N]} \quad \text{and} \quad V^M := \text{span}\{\varphi_i\}_{i \in [M]},$$
  
  where \([k] := \{1, \ldots, k\}\) for every \(k \in \mathbb{N}\).

- The set of \(s\)-sparse elements of \(U^N\)
  
  $$U^N_s := \left\{ \sum_{j \in [N]} u_j \psi_j : \|u\|_0 \leq s \right\}$$

Simplification: Let us assume the bases \(\{\psi_j\}_{j \in \mathbb{N}}\) and \(\{\varphi_q\}_{q \in \mathbb{N}}\) to be orthonormal.
Local $a$-coherence

An important tool employed in the theoretical analysis is the local $a$-coherence, a generalization of the local coherence of CS.

Definition

Given $N \in \mathbb{N} \cup \{\infty\}$, the real-valued sequence $\mu^N$ defined as

$$
\mu^N_q := \sup_{j \in [N]} |a(\psi_j, \varphi_q)|^2, \quad \forall q \in \mathbb{N},
$$

is called local $a$-coherence of $\{\psi_j\}_{j \in [N]}$ with respect to $\{\varphi_q\}_{q \in \mathbb{N}}$.

Following [F. Krahmer and R. Ward, 2014], we define a computable upper bound $\nu^N$ to $\mu^N$:

$$
\mu^N_q \leq \nu^N_q, \quad \forall q \in \mathbb{N}.
$$

Moreover, for every $M \in \mathbb{N}$, we define

$$
\nu^{N,M} := [\nu^N_1, \ldots, \nu^N_M]^T \in \mathbb{R}^M.
$$
Formalization of the CORSING procedure

PROCEDURE $\hat{u} = \text{CORSING} \ (N, \ s, \ \nu^N, \ \hat{\gamma}, \ \bar{\gamma})$

1. [Definition of $M$ and $m$]

   $$M \sim s\hat{\gamma}N; \quad m \sim s\bar{\gamma}\|\nu^{N,M}\|_1 \log(N/s);$$

2. [Test selection] Draw $\tau_1, \ldots, \tau_m$ independently at random from $[M]$ according to the probability

   $$p := \nu^{N,M}/\|\nu^{N,M}\|_1;$$

3. [Assembly] Build $A \in \mathbb{R}^{m \times N}$, $f \in \mathbb{R}^m$ and $D \in \mathbb{R}^{m \times m}$, defined as:

   $$A_{ij} := a(\psi_j, \varphi_{\tau_i}), \quad f_i := F(\varphi_{\tau_i}), \quad D_{ik} := \frac{\delta_{ik}}{\sqrt{mp_{\tau_i}}}.$$ 

4. [Recovery]

   > Find an approximate solution $\hat{u}$ to $\min_{u \in \mathbb{R}^N} \|D(Au - f)\|_2^2$, s.t. $\|u\|_0 \leq s$;

   > $\hat{u} \leftarrow \sum_{j=1}^N \hat{u}_j \psi_j.$
Main tools of the analysis

The theoretical analysis is based on three main tools:

1. the concept of **local a-coherence** between two bases;

2. **Chernoff’s bounds** for the sum of random matrices [H. Chernoff, 1952; R. Ahlswede and A. Winter, 2002; J. Tropp, 2012];

3. a variant of the classical inf-sup property, that we called **restricted inf-sup property (RISP)**, i.e.,

\[
\inf_{u \in \Sigma_s} \sup_{v \in \mathbb{R}^m} \frac{v^\top D A u}{\|u\|_2 \|v\|_2} > \tilde{\alpha} > 0,
\]

where \( \Sigma_s := \{ u \in \mathbb{R}^N : \|u\|_0 \leq s \} \).
From the $\infty$-dimensional problem to CORSING

While moving from the $\infty$-dimensional weak problem to the CORSING reduced formulation we will **track the inf-sup constant**:

<table>
<thead>
<tr>
<th></th>
<th># of tests</th>
<th>inf-sup constant</th>
</tr>
</thead>
<tbody>
<tr>
<td>Weak problem</td>
<td>$\infty$</td>
<td>$\alpha$</td>
</tr>
<tr>
<td>PG discretization</td>
<td>$M &lt; \infty$</td>
<td>$\alpha(1 - \tilde{\delta})^{\frac{1}{2}}$</td>
</tr>
<tr>
<td>CORSING</td>
<td>$m \ll M$</td>
<td>$\alpha(1 - \tilde{\delta})^{\frac{1}{2}} (1 - \overline{\delta})^{\frac{1}{2}}$</td>
</tr>
</tbody>
</table>

This will guarantee the stability of our method and will imply recovery error estimates for the CORSING technique.

\[
\inf_{u \in U_s^N} \sup_{v \in V} \frac{a(u, v)}{\|u\|_U \|v\|_V} \sim \inf_{u \in U_s^N} \sup_{v \in V^M} \frac{a(u, v)}{\|u\|_U \|v\|_V} \sim \inf_{u \in \Sigma_s^N} \sup_{v \in \mathbb{R}^m} \frac{v^\top D Au}{\|u\|_2 \|v\|_2}
\]
Uniform RISP

**Theorem**

For every $s \in \mathbb{N}$, given $\hat{\delta} \in (0, 1)$, choose $M \in \mathbb{N}$ such that

$$
\sum_{q > M} \mu_q^N \leq \frac{\alpha^2 \hat{\delta}}{s}.
$$

Then, for every $\varepsilon > 0$ and $\bar{\delta} \in (0, 1)$, provided

$$
m \gtrsim \bar{\delta}^{-2} \| \nu^{N,M} \|_1 [s^2 \log(eN/s) + s \log(s/\varepsilon)],
$$

the following uniform RISP holds with probability $\geq 1 - \varepsilon$

$$
\inf_{u \in \Sigma_s^N} \sup_{v \in \mathbb{R}^m} \frac{v^\top D A u}{\|u\|_2 \|v\|_2} > \tilde{\alpha} > 0,
$$

where $\tilde{\alpha} := (1 - \hat{\delta})^{\frac{1}{2}} (1 - \bar{\delta})^{\frac{1}{2}} \alpha$. 
Non-uniform RISP: sketch of the proof (1/2)

The proof can be organized as follows:

1. Fix $S \subseteq [N]$, with $|S| = s$, and notice that

$$\inf_{u \in \mathbb{R}^s} \sup_{v \in \mathbb{R}^m} \frac{v^\top D A_S u}{\|u\|_2 \|v\|_2} = \left[ \lambda_{\min}(A_S^\top D^2 A_S) \right]^{\frac{1}{2}} = \left[ \lambda_{\min}(\overline{X}) \right]^{\frac{1}{2}}.$$

Indeed, $A_S^\top D^2 A_S$ is the sample mean of random matrices

$$(A_S^\top D^2 A_S)_{jk} = \frac{1}{m} \sum_{i=1}^{m} \frac{1}{p_{\tau_i}} a(\psi_j, \phi_{\tau_i}) a(\psi_k, \phi_{\tau_i}).$$

$$=: X_{jk}^{\tau_i}$$

2. The minimum eigenvalue of $X_{\tau_i}$ can be controlled in expectation:

$$\sum_{q>M} \mu_q^N \leq \frac{\hat{\delta} \alpha^2}{s} \implies \lambda_{\min}(\mathbb{E}[X_{\tau_i}])^{\frac{1}{2}} = \inf_{u \in U_S^N, v \in V^M} \sup_{u \in U_S^N} \frac{a(u, v)}{\|u\|_U \|v\|_V} \geq (1-\hat{\delta})^{\frac{1}{2}} \alpha$$

3. The thesis is proved by resorting to the matrix Chernoff bounds.
Non-uniform RISP: sketch of the proof (2/2)

Theorem (Matrix Chernoff bounds)

Consider a finite sequence of i.i.d. random, symmetric $s \times s$ real matrices $M^1, \ldots, M^m$ such that

$$0 \leq \lambda_{\min}(M^i) \text{ and } \lambda_{\max}(M^i) \leq R \quad \text{almost surely, } \forall i \in [m].$$

Define $\overline{M} := \frac{1}{m} \sum_{i=1}^{m} M^i$ and $\lambda_* := \lambda_{\min}(\mathbb{E}[M^i])$. Then,

$$\mathbb{P}\{\lambda_{\min}(\overline{M}) \leq (1 - \delta)\lambda_*\} \lesssim s \exp \left( - \frac{m\delta^2\lambda_*}{R} \right), \quad \forall \delta \in [0, 1].$$

- After choosing $M^i = X^{\tau_i}$, direct computations show that

$$0 \leq \lambda_{\min}(X^{\tau_i}) \quad \text{and} \quad \lambda_{\max}(X^{\tau_i}) \leq s\|\nu_{N,M}^{N_1}\|_1.$$

- Finally, we consider the inf-sup over $U_{s^N}$ employing a union bound.
Recovery error analysis

Our aim is to compare the recovery error $\|\hat{u} - u\|_U$ with the best $s$-term approximation error of the exact solution $u$ in $U^N$, i.e. the quantity $\|u^s - u\|_U$, where

$$u^s := \arg \min_{w \in U^N_s} \|w - u\|_U.$$ 

A key quantity is the following preconditioned random residual

$$\mathcal{R}(u^s) := \left[ \frac{1}{m} \sum_{i=1}^{m} \frac{1}{p_{\tau_i}} \left[ a(u^s, \varphi_{\tau_i}) - \mathcal{F}(\varphi_{\tau_i}) \right]^2 \right]^{\frac{1}{2}} = \|D(Au^s - f)\|_2.$$ 

**Assumption:** we assume that $\hat{u}$ solves the problem

$$\min_{u \in \mathbb{R}^N} \|D(Au - f)\|_2^2, \text{ s.t. } \|u\|_0 \leq s$$

exactly (even if, in reality, OMP can only approximate its solution).
Two lemmas about $\mathcal{R}(u^s)$

◆ An argument analogous to Cea’s lemma shows the following

**Lemma**

*If the following uniform $2s$-sparse RISP holds*

$$\inf_{u \in \Sigma^{N}_{2s}} \sup_{v \in \mathbb{R}^m} \frac{v^\top D A u}{\|u\|_2 \|v\|_2} > \tilde{\alpha} > 0,$$

*then the CORSING procedure computes a solution $\hat{u}$ such that*

$$\|\hat{u} - u^s\|_U < \frac{2}{\tilde{\alpha}} \mathcal{R}(u^s).$$

◆ Moreover, this mysterious residual behaves nicely *in expectation!*

**Lemma**

$$\mathbb{E}[\mathcal{R}(u^s)^2] \leq \beta^2 \|u^s - u\|_U^2,$$

*where $\beta$ is the continuity constant of $a(\cdot, \cdot)$.***
Error estimate in expectation

Theorem (CORSING recovery in expectation)

Let \( s \leq N \) and \( \mathcal{K} > 0 \) be such that \( \|u\|_U \leq \mathcal{K} \) and \( \hat{\delta}, \tilde{\delta} \in (0, 1) \). Choose \( M \in \mathbb{N} \) such that the following truncation condition is fulfilled

\[
\sum_{q > M} \mu_q^N \leq \frac{\alpha^2 \hat{\delta}}{s}.
\]

Then, for every \( \varepsilon > 0 \), provided

\[
m \gtrsim \tilde{\delta}^{-2} \|\nu^{N,M}\|_1 \left[ s^2 \log(N/s) + s \log(s/\varepsilon) \right],
\]

the truncated CORSING solution \( \mathcal{T}_K \hat{u} \) fulfills

\[
\mathbb{E}[\|\mathcal{T}_K \hat{u} - u\|_U] \leq \left( 1 + \frac{2\beta}{\tilde{\alpha}} \right) \|u^s - u\|_U + 2\mathcal{K}\varepsilon,
\]

where \( \tilde{\alpha} = (1 - \hat{\delta})^{1/2} (1 - \tilde{\delta})^{1/2} \alpha \) and \( \mathcal{T}_K(w) := \max(1, \mathcal{K}/\|w\|_U)w \).

Remarks:

- A possible choice for \( \mathcal{K} \) is \( \|\mathcal{F}\|_{V^*}/\alpha \).
- An analogous result holds in probability.
Proposition (CORSING $\mathcal{HS}$ recovery)

Fix a maximum hierarchical level $L \in \mathbb{N}$, corresponding to $N = 2^{L+1} - 1$. Then, for every $\varepsilon \in (0, 2^{-1/3}]$ and $s \leq 2N/e$, provided

$$M \gtrsim sN, \quad m \gtrsim \log M[s^2 \log(N/s) + s \log(s/\varepsilon)]$$

and chosen the upper bound $\nu^N$ as

$$\nu_q^N \sim \frac{1}{q}, \quad \forall q \in \mathbb{N},$$

the CORSING $\mathcal{HS}$ solution to the homogeneous 1D Poisson problem fulfills

$$\mathbb{E}[|\mathcal{T}_K \hat{u} - u|_{H^1}] \leq 5|u^s - u|_{H^1} + 2K\varepsilon,$$

for every $K > 0$ such that $|u|_{H^1} \leq K$. 
Sketch of the proof

For the 1D Poisson problem we have the following bound

\[ \mu_q^N \lesssim \min \left\{ \frac{N}{q^2}, \frac{1}{q} \right\}. \]

Then, we have

\[ \sum_{q > M} \mu_q^N \lesssim N \sum_{q > M} \frac{1}{q^2} \sim \frac{N}{M}, \] required to be \( \lesssim \frac{1}{s}. \)

Moreover, choosing \( \nu_q^N \sim 1/q \) yields

\[ \| \nu^{N,M} \|_1 \sim \sum_{q=1}^{M} \frac{1}{q} \sim \log M. \]
Application to 1D ADR problems

Consider the problem

\[
\text{find } u \in H_0^1(\Omega) : (u', v') + b(u', v) + \rho(u, v) = (f, v), \quad \forall v \in H_0^1(\Omega) \tag{ADR}
\]

with \( b, \rho \in \mathbb{R}, \rho > 0 \) and \( \Omega = (0, 1) \). Let \( H_0^1(\Omega) \) be endowed with \( | \cdot |_{H^1(\Omega)} \).

Proposition (CORSING \( \mathcal{HS} \) for 1D ADR)

Fix \( N \in \mathbb{N} \). Then, for every \( \varepsilon \in (0, 2^{-1/3}] \) and \( s \leq 2N/e \), provided that

\[
M \gtrsim sN, \quad |b|/M \lesssim 1, \quad |\rho|/M^2 \lesssim 1,
\]

\[
m \gtrsim (\log M + |b|^2 + |\rho|^2) [s^2 \log(N/s) + s \log(s/\varepsilon)],
\]

and chosen the upper bound \( \nu_N \) such that

\[
\nu_q^N \sim \frac{1}{q} + \frac{|b|^2}{q^3} + \frac{|\rho|^2}{q^5}, \quad \forall q \in \mathbb{N},
\]

the CORSING \( \mathcal{HS} \) solution to (ADR) fulfills

\[
\mathbb{E} \left[ |\mathcal{T}_K \hat{u} - u|_{H^1(\Omega)} \right] \lesssim (1 + |b| + |\rho|)|u^s - u|_{H^1(\Omega)} + \mathcal{K} \varepsilon,
\]

for every \( \mathcal{K} > 0 \) such that \( |u|_{H^1(\Omega)} \leq \mathcal{K} \).
Application to the 1D diffusion equation

Let $\Omega = (0, 1)$ and consider the problem

$$\text{find } u \in H^1_0(\Omega) : \quad (\eta u', v') = (f, v), \quad \forall v \in H^1_0(\Omega).$$

(DIF)

Proposition

Let $\eta \in L^\infty(\Omega)$ be such that

1. there exists $\eta_{\text{min}} > 0$ so that $\eta(x) \geq \eta_{\text{min}}$, for almost every $x \in \Omega$;
2. there exists a finite set $\mathcal{P} \subseteq \overline{\Omega}$ such that $\eta \in C^2(\Omega \setminus \mathcal{P})$;
3. $\sup_{x \in \Omega \setminus \mathcal{P}} |\eta^{(k)}(x)| < \infty$, for $k = 1, 2$.

Fix $L \in \mathbb{N}$ and put $N = 2^{L+1} - 1$. Then, provided

$$\nu_q^N \sim 1/q, \quad \forall q \in \mathbb{N},$$

and

$$M \gtrsim sN, \quad m \gtrsim \log M[s^2 \log(N/s) + s \log(s/\varepsilon)],$$

the CORSING $\mathcal{HS}$ solution $\hat{u}$ to (DIF) fulfills

$$\mathbb{E}[|\mathcal{T}_K \hat{u} - u|_{H^1(\Omega)}] \leq \left(1 + \frac{4\|\eta\|_{L^\infty}}{\eta_{\text{min}}}\right)|u^s - u|_{H^1(\Omega)} + 2K\varepsilon,$$

for every $K > 0$ such that $|u|_{H^1(\Omega)} \leq K$. 

43
A RIP theorem for CORSING

(with S. Dirksen, H.C. Jung, H. Rauhut, RWTH Aachen)

Theorem (RIP for CORSING)

Let $s, N \in \mathbb{N}$, with $s < N$, and $\hat{\delta} \in (0, 1)$. Suppose the truncation condition

$$\sum_{q > M} \mu_q^N \leq \frac{\alpha^2 \hat{\delta}}{s}. \tag{1}$$

to be fulfilled. Then, provided $\delta \in (1 - (1 - \hat{\delta}) \frac{\alpha^2}{\beta^2}, 1)$, and

$$m \gtrsim \delta^{-2} \|\nu^{N,M}\|_1 s \log^3(s) \log(N), \tag{2}$$

it holds

$$\mathbb{P}\{\beta^{-1} DA \in RIP(s, \delta)\} \geq 1 - N^{-\log^3(s)}, \tag{3}$$

where $\beta$ is the continuity constant of $a(\cdot, \cdot)$.  

CORSING computes the best $s$-term approximation to $u$ in $O(smN)$ flops.
Further results

- The previous results hold in the case of nonorthogonal trial and test functions. Indeed, they suffice to be Riesz bases, i.e.,

\[
\| \sum_{j \in \mathbb{N}} u_j \psi_j \|_U \sim \| u \|_2, \quad \forall u \in U^N.
\]

- We checked the theoretical hypotheses on the local $a$-coherence for the 2D and 3D ADR equations numerically.

**Figure:** The plot shows that

\[
\nu^N_q \sim \frac{1}{q_1 q_2 q_3}
\]

is a local $a$-coherence upper bound for the 3D Poisson problem (CORSING QS).
Wrap up: main results

✓ CS can be successfully applied to solve PDEs, such as 1D, 2D, and 3D ADR problems, or the 2D Stokes problem;

✓ CORSING can considerably reduce the computational cost associated with a full-PG discretization;

✓ the local a-coherence is crucial to understand the behavior of the method theoretically;

Future directions

► Speed-up the recovery phase (get rid of the “N” in the cost $O(smN)$);

► Investigate other trial/test combinations: e.g., biorthogonal wavelets, instead of hierarchical basis (ongoing);

► 2D and 3D theory (ongoing);

► apply CORSING to more challenging benchmarks, such as Navier-Stokes, or nonlocal problems;

► adapt the CORSING technique to the case of parametric PDEs.
Thank you for your attention!

...questions?