Periodic Homogenization For Elliptic Nonlocal Equations
(PIMS Workshop on Analysis of nonlinear PDEs and free boundary problems)

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The Set-Up

Family of Oscillatory Nonlocal Equations:

\[ \begin{aligned}
F(u^\varepsilon, \frac{x}{\varepsilon}) &= 0 \quad \text{in } D \\
u^\varepsilon &= g \quad \text{on } \mathbb{R}^n \setminus D
\end{aligned} \]

Translation Invariant Limit Nonlocal Equations

\[ \begin{aligned}
\bar{F}(\bar{u}, x) &= 0 \quad \text{in } D \\
\bar{u} &= g \quad \text{on } \mathbb{R}^n \setminus D.
\end{aligned} \]

GOAL

Prove there is a unique nonlocal operator \( \bar{F} \) so that \( u^\varepsilon \) will be very close to \( \bar{u} \) as \( \varepsilon \to 0 \). (Homogenization takes place.)
The Set-Up

Family of Oscillatory Nonlocal Equations:

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\begin{cases}
    F(u^\varepsilon, \frac{x}{\varepsilon}) = 0 & \text{in } D \\
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\end{cases}
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Translation Invariant Limit Nonlocal Equations

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    \bar{F}(\bar{u}, x) = 0 & \text{in } D \\
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F(u^\varepsilon, \frac{x}{\varepsilon}) &= 0 \quad \text{in } D \\
\varepsilon u^\varepsilon &= g \quad \text{on } \mathbb{R}^n \setminus D
\end{aligned}
\]

\[
F(u, \frac{x}{\varepsilon}) = \\
\inf_{\alpha} \sup_{\beta} \left\{ f_{\alpha\beta}(\frac{x}{\varepsilon}) + \int_{\mathbb{R}^n} (u(x + y) + u(x - y) - 2u(x))K_{\alpha\beta}(\frac{x}{\varepsilon}, y)dy \right\}.
\]

Think of a more familiar **2nd order** equation:

\[
F(D^2 u, \frac{x}{\varepsilon}) = \inf_{\alpha} \sup_{\beta} \left\{ f_{\alpha\beta}(\frac{x}{\varepsilon}) + a_{ij}^{\alpha\beta}(\frac{x}{\varepsilon})u_{x_i x_j}(x) \right\}
\]
The Set-Up

Periodic Nonlocal Operator $G$

for all $z \in \mathbb{Z}^n$

$$G(u, x + z) = G(u(\cdot + z), x)$$

Our $F$ will be periodic when $f^{\alpha\beta}$ and $K^{\alpha\beta}$ are periodic in $x$.

Translation Invariant Nonlocal Operator $G$

$G$ is translation invariant if for any $y \in \mathbb{R}^n$,

$$G(u, x + y) = G(u(\cdot + y), x).$$
Main Theorem

Theorem (S. ‘08; Homogenization of Nonlocal Equations)

If $F$ is periodic and uniformly elliptic, plus technical assumptions, then there exists a translation invariant elliptic nonlocal operator $\bar{F}$ with the same ellipticity as $F$, such that $u^\varepsilon \to \bar{u}$ locally uniformly and $\bar{u}$ is the unique solution of

\[
\begin{align*}
\bar{F}(\bar{u}, x) &= 0 \quad \text{in } D \\
\bar{u} &= g \quad \text{on } \mathbb{R}^n \setminus D.
\end{align*}
\]
Interpretations and Applications

- **Linear Case**— Determine effective dynamics of Lévy Process in inhomogeneous media

\[
f\left(\frac{X}{\varepsilon}\right) + \int_{\mathbb{R}^n} (u(x + y) + u(x - y) - 2u(x))K\left(\frac{X}{\varepsilon}, y\right)dy
\]

- **Optimal Control Case**— Determine an effective optimal cost of control of Lévy Processes in inhomogeneous media

\[
\inf_{\alpha} \left\{ f^{\alpha}\left(\frac{X}{\varepsilon}\right) + \int_{\mathbb{R}^n} (u(x + y) + u(x - y) - 2u(x))K^{\alpha}\left(\frac{X}{\varepsilon}, y\right)dy \right\}
\]

- **Two Player Game Case**— Determine an effective value of a two player game of a Lévy Process in inhomogeneous media

\[
\inf_{\alpha} \sup_{\beta} \left\{ f^{\alpha\beta}\left(\frac{X}{\varepsilon}\right) + \int_{\mathbb{R}^n} (u(x + y) + u(x - y) - 2u(x))K^{\alpha\beta}\left(\frac{X}{\varepsilon}, y\right)dy \right\}
\]
The Set-Up– Assumptions on $F$

“Ellipticity”

\[
\frac{\lambda}{|y|^{n+\sigma}} \leq K^{\alpha\beta}(x, y) \leq \frac{\Lambda}{|y|^{n+\sigma}}
\]

Scaling

\[
K^{\alpha\beta}(x, \lambda y) = \lambda^{-n-\sigma} K^{\alpha\beta}(x, y).
\]

Symmetry

\[
K^{\alpha\beta}(x, -y) = K^{\alpha\beta}(x, y)
\]
## Recent Background– Nonlocal Elliptic Equations

### Existence/Uniqueness (Barles-Chasseigne-Imbert)

Given basic assumptions on $K^{\alpha\beta}$ and $f^{\alpha\beta}$, there exist unique solutions to the Dirichlet Problems $F(u^\varepsilon, x/\varepsilon) = 0, \bar{F}(\bar{u}, x) = 0$.

### Regularity (Silvestre, Caffarelli-Silvestre)

$u^\varepsilon$ are Hölder continuous, depending only on $\lambda, \Lambda, \|f^{\alpha\beta}\|_\infty$, dimension, and $g$. (In particular, continuous uniformly in $\varepsilon$.)

### Nonlocal Ellipticity (Caffarelli-Silvestre)

If $u$ and $v$ are $C^{1,1}$ at a point, $x$, then

$$M^-(u - v)(x) \leq F(u, x) - F(v, x) \leq M^+(u - v)(x).$$

$$M^- u(x) = \inf_{\alpha\beta} \left\{ L^{\alpha\beta} u(x) \right\} \quad \text{and} \quad M^+ u(x) = \sup_{\alpha\beta} \left\{ L^{\alpha\beta} u(x) \right\}.$$
Recent Background– 2nd Order Homogenization

The “Corrector” Equation (Caffarelli-Souganidis-Wang)

For each matrix, $Q$, fixed, $\bar{F}(Q)$ is the unique constant such that the solutions, $v^\varepsilon$, of

$$\begin{cases} F(Q + D^2v^\varepsilon, \frac{x}{\varepsilon}) = \bar{F}(Q) \text{ in } B_1 \\ v^\varepsilon(x) = 0 \text{ on } \partial B_1, \end{cases}$$

satisfy the decay property as $\varepsilon \to 0$, $\|v^\varepsilon\|_\infty \to 0$.

This generalizes the notion of the True Corrector Equation (Lions-Papanicolaou-Varadhan, 1st order HJE)

$\bar{F}(Q)$ is the unique constant such that there is a global periodic solution of

$$F(Q + D^2v, y) = \bar{F}(Q) \text{ in } \mathbb{R}^n.$$
Recent Background– 2nd Order Homogenization

The “Corrector” Equation (Caffarelli-Souganidis-Wang)

For each matrix, $Q$, fixed, $\bar{F}(Q)$ is the **unique** constant such that the solutions, $v^\varepsilon$, of

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$$

satisfy the **decay property** as $\varepsilon \to 0$, $\|v^\varepsilon\|_\infty \to 0$.

This generalizes the notion of the

**True Corrector Equation** (Lions-Papanicolaou-Varadhan, 1st order HJE)

$\bar{F}(Q)$ is the unique constant such that there is a global periodic solution of

$$F(Q + D^2 v, y) = \bar{F}(Q) \text{ in } \mathbb{R}^n.$$
Perturbed Test Function Method

Need to Determine Effective operator

⇒ All information is in original operator

$F(\cdot, x/\varepsilon) = 0$

Can we perturb $\phi$ to $\phi + \nu^\varepsilon$
to COMPARE WITH $u^\varepsilon$???

$\bar{F}(\phi, x_0) \geq 0$

⇒ $F(\phi + \nu^\varepsilon, x/\varepsilon) = \bar{F} \geq 0$

To go BACK from comparison of $\phi + \nu^\varepsilon$ and $u^\varepsilon$
TO comparison of $\phi$ and $u$ NEED

$|\nu^\varepsilon| \to 0$ as $\varepsilon \to 0$
Perturbed Test Function Method

Need to Determine Effective operator

\[ F(\cdot, x/\varepsilon) = 0 \]

All information is in original operator

Can we perturb \( \phi \) to \( \phi + v^\varepsilon \) to COMPARE WITH \( u^\varepsilon \)???

\[ F(\phi + v^\varepsilon, x/\varepsilon) = \bar{F} \geq 0 \]

To go BACK from comparison of \( \phi + v^\varepsilon \) and \( u^\varepsilon \) TO comparison of \( \phi \) and \( u \) NEED

\[ |v^\varepsilon| \to 0 \text{ as } \varepsilon \to 0 \]
Strategy

- Most of the arguments for 2nd order homogenization are based on \textsc{comparision} + \textsc{regularity}
- Nonlocal equations have good \textsc{comparision} + \textsc{regularity} properties

$\implies$ We should try to modify techniques of the 2nd order setting to the nonlocal setting
Difficulties Taking Ideas to Nonlocal Setting

- The space of test functions is much larger! $C^2_b(\mathbb{R}^n)$ versus $S^n$
- Test function space is not invariant under the scaling of the operators $u \mapsto \varepsilon^\sigma u(\cdot/\varepsilon)$
- $\bar{F}(\phi, \cdot)$ is a function, not a constant
- What should be the “corrector” equation? We can’t just “freeze” the hessian, $D^2\phi(x_0)$, at a point $x_0$
Bad Test Function Scaling, But Good $F$ Scaling

\[
L^{\alpha\beta}[\varepsilon^\sigma u(\cdot \varepsilon)](x) = L^{\alpha\beta}[u](\varepsilon x)
\]

\[
L^{\alpha\beta}u(x) = \int_{\mathbb{R}^n} (u(x + y) + u(x - y) - 2u(x))K^{\alpha\beta}(\varepsilon x, y)dy
\]

Put The Test Function Inside

\[
\begin{cases}
F(\phi + \nu^\varepsilon, \varepsilon x) = \mu & \text{in } B_1 \\
\nu^\varepsilon(x) = 0 & \text{on } \mathbb{R}^n \setminus B_1,
\end{cases}
\]
Scaling Test Functions?

Bad Test Function Scaling, But Good $F$ Scaling

\[
L^{\alpha\beta}[\varepsilon^{\sigma} u(\cdot)](x) = L^{\alpha\beta}[u](\frac{x}{\varepsilon})
\]

\[
L^{\alpha\beta} u(x) = \int_{\mathbb{R}^n} (u(x + y) + u(x - y) - 2u(x)) K^{\alpha\beta}(\frac{x}{\varepsilon}, y) dy
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Put The Test Function Inside

\[
\begin{cases}
F(\phi + v^\varepsilon, \frac{x}{\varepsilon}) = \mu \quad \text{in } B_1 \\
v^\varepsilon(x) = 0 \quad \text{on } \mathbb{R}^n \setminus B_1,
\end{cases}
\]
“Corrector” Equation

Equation for $\phi + v^\varepsilon$

\[ F(\phi + v^\varepsilon, \frac{x}{\varepsilon}) = \]
\[ \inf_{\alpha} \sup_{\beta} \left\{ f^{\alpha\beta}(\frac{x}{\varepsilon}) + \int_{\mathbb{R}^n} (\phi(x + y) + \phi(x - y) - 2\phi(x))K^{\alpha\beta}(\frac{x}{\varepsilon}, y)dy \right\} \]
\[ + \int_{\mathbb{R}^n} (v^\varepsilon(x + y) + v^\varepsilon(x - y) - 2v^\varepsilon(x))K^{\alpha\beta}(\frac{x}{\varepsilon}, y)dy \}

“Frozen” operator on $\phi$ at $x_0$

\[ [L^{\alpha\beta}\phi(x_0)](x) = \int_{\mathbb{R}^n} (\phi(x_0 + z) + \phi(x_0 - z) - 2\phi(x_0))K^{\alpha\beta}(x, z)dz \]
“Corrector” Equation

equation for $\phi + v^\varepsilon$

$$F(\phi + v^\varepsilon, \frac{x}{\varepsilon}) =$$

$$\inf_{\alpha} \sup_{\beta} \left\{ f^{\alpha\beta}(\frac{x}{\varepsilon}) + \int_{\mathbb{R}^n} (\phi(x + y) + \phi(x - y) - 2\phi(x))K^{\alpha\beta}(\frac{x}{\varepsilon}, y)dy \right\}$$

“frozen” operator on $\phi$ at $x_0$

$$[L^{\alpha\beta}\phi(x_0)](x) = \int_{\mathbb{R}^n} (\phi(x_0 + z) + \phi(x_0 - z) - 2\phi(x_0))K^{\alpha\beta}(x, z)dz$$
“Corrector” Equation

Analogy to 2nd order equation

\[
\begin{align*}
a_{ij}\left(\frac{x}{\varepsilon}\right)(\phi + v)_{x_ix_j}(x) &= a_{ij}\left(\frac{x}{\varepsilon}\right)\phi_{x_ix_j}(x) + a_{ij}\left(\frac{x}{\varepsilon}\right)v_{x_ix_j}(x) \\
\text{and } a_{ij}\left(\frac{x}{\varepsilon}\right)\phi_{x_ix_j}(x) \text{ is uniformly continuous in } x.
\end{align*}
\]

Free and frozen variables, \(x\) and \(x_0\)

Uniform continuity (Caffarelli-Silvestre)

\([L^{\alpha\beta}\phi(x_0)](x)\) is uniformly continuous in \(x_0\), independent of \(x\) and \(\alpha\beta\)
NEW OPERATOR $F_{\phi,x_0}$

$$F_{\phi,x_0}(v^\epsilon, \frac{x}{\epsilon}) = \inf_{\alpha} \sup_{\beta} \{ f^{\alpha\beta}(\frac{x}{\epsilon}) + [L^{\alpha\beta} \phi(x_0)](\frac{x}{\epsilon})$$

$$+ \int_{\mathbb{R}^n} (v^\epsilon(x + y) + v^\epsilon(x - y) - 2v^\epsilon(x)) K^{\alpha\beta}(\frac{x}{\epsilon}, y)dy \}$$

New “Corrector” Equation

$$\begin{cases} F_{\phi,x_0}(v^\epsilon, \frac{x}{\epsilon}) = \bar{F}(\phi, x_0) \quad \text{in } B_1(x_0) \\ v^\epsilon = 0 \quad \text{on } \mathbb{R}^n \setminus B_1(x_0). \end{cases}$$
Proposition (S. '08; “Corrector” Equation)

There exists a unique choice for the value of $\bar{F}(\phi, x_0)$ such that the solutions of the “corrector” equation also satisfy

$$\lim_{\varepsilon \to 0} \max_{B_1(x_0)} |v^\varepsilon| = 0.$$ 

(via the perturbed test function method, this proposition is equivalent to homogenization)
(Caffarelli-Sougandis-Wang... *In spirit)

Consider a generic choice of a Right Hand Side, \( l \) is fixed

\[
\begin{cases}
F_{\phi,x_0}(v_{l}^{\varepsilon}, \frac{x}{\varepsilon}) = l & \text{in } B_1(x_0) \\
v_{l}^{\varepsilon} = 0 & \text{on } \mathbb{R}^n \setminus B_1(x_0).
\end{cases}
\]

How does the choice of \( l \) affect the decay of \( v_{l}^{\varepsilon} \)?

**Decay property**

\[
\lim_{\varepsilon \to 0} \max_{B_1(x_0)} |v^{\varepsilon}| = 0 \iff (v_{l}^{\varepsilon})^* = (v_{l}^{\varepsilon})_* = 0
\]
Finding $\bar{F}$... Variational Problem

(Caffarelli-Sougandis-Wang... *In spirit)

Consider a generic choice of a Right Hand Side, $l$ is fixed

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\begin{aligned}
F_{\phi,x_0}(v^\varepsilon, x^\varepsilon) &= l & \text{in } B_1(x_0) \\
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$$

How does the choice of $l$ affect the decay of $v^\varepsilon$?

**decay property**

$$
\lim_{\varepsilon \to 0} \max_{B_1(x_0)} |v^\varepsilon| = 0 \iff (v^\varepsilon)^* = (v^\varepsilon)_* = 0
$$
$I$ very negative

$p^+(x) = (1 - |x|^2)^2 \cdot 1_{B_1}$ is a subsolution of equation $\implies (v_\varepsilon^*) > 0$ and we missed the goal.

$$\left( F_{\phi, x_0}(v_\varepsilon^*, \frac{x}{\varepsilon}) = I \right)$$
Variational Problem

\[ \left( F_{\phi,x_0}(v_\varepsilon, \frac{x}{\varepsilon}) = l \right) \]

/ very positive

\[ p^-(x) = -(|x|^2 - 1)^2 \cdot \mathbb{1}_{B_1} \] is a supersolution of equation

\[ \implies (v_\varepsilon)^* < 0 \text{ and we missed the goal,} \]

Can we choose an \( l \) in the middle that is “JUST RIGHT”?
Variational Problem

\[ (F_{\phi,x_0}(\nu^\varepsilon, \frac{x}{\varepsilon}) = l) \]

\[ p^-(x) = -\left(|x|^2 - 1\right)^2 \cdot 1_{B_1} \] is a \textbf{supersolution} of equation

\[ \implies (\nu^\varepsilon)^* < 0 \text{ and we missed the goal, but in the other direction.} \]

Can we choose an \( l \) in the middle that is “JUST RIGHT”?
Variational Problem

\[
\left( F_{\phi,x_0}(\nu_{i \epsilon}, \frac{x}{\epsilon}) = l \right)
\]

/ very positive

\[ p^-(x) = -(|x|^2 - 1)^2 \cdot \mathbb{1}_{B_1} \] is a supersolution of equation

\[ \implies (\nu^\epsilon_i)^* < 0 \text{ and we missed the goal, but in the other direction.} \]

Can we choose an \( \ell \) in the middle that is “JUST RIGHT”?
Variational Problem

Can we choose an / in the middle that is “JUST RIGHT”?
Obstacle Problem

(Caffarelli-Sougandis-Wang) The answer is YES.

Information From Obstacle Problem

The obstacle problem gives relationship between the choice of $l$ and the decay of $v^\varepsilon_l$. 
## Obstacle Problem

**The Solution of The Obstacle Problem In a Set A**

\[ U_A^l = \inf \{ u : F_{\phi,x_0}(u, y) \leq l \text{ in } A \text{ and } u \geq 0 \text{ in } \mathbb{R}^n \} \]

**equation:** \( U_A^l \) is the least supersolution of \( F_{\phi,x_0} = l \) in \( A \)

**obstacle:** \( U_A^l \) must be above the obstacle which is 0 in all of \( \mathbb{R}^n \)

### Lemma (Hölder Continuity)

\( U_A^l \) is \( \gamma \)-Hölder Continuous depending only on \( \lambda, \Lambda, \| f^{\alpha\beta} \|_{\infty}, \phi, \) dimension, and \( A \).

### Monotonicity and Periodicity of Obstacle Problem

If \( A \subset B \), then \( U_A^l \leq U_B^l \). For \( z \in \mathbb{Z}^n \), \( U_{A+z}^l(x) = U_A^l(x - z) \)
## Obstacle Problem

### The Solution of The Obstacle Problem In a Set $A$

\[ U_A^l = \inf \{ u : F_{\phi,x_0}(u, y) \leq l \text{ in } A \text{ and } u \geq 0 \text{ in } \mathbb{R}^n \} \]

**equation:** $U_A^l$ is the least supersolution of $F_{\phi,x_0} = l$ in $A$

**obstacle:** $U_A^l$ must be above the obstacle which is 0 in all of $\mathbb{R}^n$

### Lemma (Hölder Continuity)

$U_A^l$ is $\gamma$-Hölder Continuous depending only on $\lambda$, $\Lambda$, $\|f^{\alpha\beta}\|_\infty$, $\phi$, dimension, and $A$.

### Monotonicity and Periodicity of Obstacle Problem

If $A \subset B$, then $U_A^l \leq U_B^l$. For $z \in \mathbb{Z}^n$, $U_{A+z}^l(x) = U_A^l(x - z)$
Obstacle Problem

NOTATION

Rescaled Solution

\[ u^{\varepsilon, l} = \inf \left\{ u : F_{\phi, x_0}(u, \frac{y}{\varepsilon}) \leq l \text{ in } Q_1 \text{ and } u \geq 0 \text{ in } \mathbb{R}^n \right\}. \]

Solution in \( Q_1 \) and Solution in \( Q_1/\varepsilon \)

\[ u^{\varepsilon, l}(x) = \varepsilon^\sigma U_{Q_1/\varepsilon}^l \left( \frac{x}{\varepsilon} \right) \]
Obstacle Problem

Dichotomy

(i) For all $\varepsilon > 0$, $U_{Q_1/\varepsilon}^I = 0$ for at least one point in every complete cell of $\mathbb{Z}^n$ contained in $Q_1/\varepsilon$.

(ii) There exists some $\varepsilon_0$ and some cell, $C_0$, of $\mathbb{Z}^n$ such that $U_{Q_1/\varepsilon_0}^I (y) > 0$ for all $y \in C_0$.

Lemma (Part (i) of The Dichotomy)

If (i) occurs, then $(v^\varepsilon_i)^* \leq 0$.

Lemma (Part (ii) of The Dichotomy)

If (ii) occurs, then $(v^\varepsilon_i)^* \geq 0$.
### Dichotomy

(i) For all $\varepsilon > 0$, $U_{Q_1/\varepsilon}^l = 0$ for at least one point in every complete cell of $\mathbb{Z}^n$ contained in $Q_1/\varepsilon$.

(ii) There exists some $\varepsilon_0$ and some cell, $C_0$, of $\mathbb{Z}^n$ such that $U_{Q_1/\varepsilon_0}^l(y) > 0$ for all $y \in C_0$.

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#### Lemma (Part (i) of The Dichotomy)

*If (i) occurs, then $(v_l^\varepsilon)^* \leq 0$.*

#### Lemma (Part (ii) of The Dichotomy)

*If (ii) occurs, then $(v_l^\varepsilon)^* \geq 0$.*
Obstacle Problem

**Dichotomy**

(i) For all $\varepsilon > 0$, $U^l_{Q_1/\varepsilon} = 0$ for at least one point in every complete cell of $\mathbb{Z}^n$ contained in $Q_{1/\varepsilon}$.

(ii) There exists some $\varepsilon_0$ and some cell, $C_0$, of $\mathbb{Z}^n$ such that $U^l_{Q_{1/\varepsilon_0}} (y) > 0$ for all $y \in C_0$.

**Lemma (Part (i) of The Dichotomy)**

If (i) occurs, then $(v^\varepsilon_i)^* \leq 0$.

**Lemma (Part (ii) of The Dichotomy)**

If (ii) occurs, then $(v^\varepsilon_i)^* \geq 0$.
## Obstacle Problem

### Dichotomy

(i) For all $\varepsilon > 0$, $U_{Q_1/\varepsilon}^l = 0$ for at least one point in **every** complete cell of $\mathbb{Z}^n$ contained in $Q_{1/\varepsilon}$.

(ii) There exists some $\varepsilon_0$ and some cell, $C_0$, of $\mathbb{Z}^n$ such that $U_{Q_1/\varepsilon_0}^l (y) > 0$ for all $y \in C_0$.

### Lemma (Part (i) of The Dichotomy)

*If (i) occurs, then $(v_1^\varepsilon)^* \leq 0$.*

### Lemma (Part (ii) of The Dichotomy)

*If (ii) occurs, then $(v_1^\varepsilon)^* \geq 0$.*
Obstacle Problem

Proof of First Lemma (If (i) occurs, then $(v^\varepsilon_i)^* \leq 0$)... 

- Rescale back to $Q_1$.
  Definition of $u^{\varepsilon, l} \implies v^\varepsilon_i \leq u^{\varepsilon, l}$
- (i) $\implies u^{\varepsilon, l} = 0$ at least once in EVERY cell of $\varepsilon \mathbb{Z}^n$. Hölder Continuity $\implies u^{\varepsilon, l} \leq C\varepsilon^\gamma$. 

![Graph showing the relationship between $u_\varepsilon$ and $v_\varepsilon$ within the domain $Q_1$. The graph illustrates the boundedness of $u_\varepsilon$ and $v_\varepsilon$ with $C\varepsilon^\gamma$.](attachment:graph.png)
Obstacle Problem

Proof of Second Lemma (If (ii) occurs, then \((v_i^\varepsilon)_* \geq 0)\)...

- Given any \(\delta > 0\), Periodicity, Monotonicity, and (ii) allow construction of a connected cube \(C_\varepsilon \subset Q_1\) such that \(u^{\varepsilon, l} > 0\) in \(C_\varepsilon\) and \(|C_\varepsilon| / |Q_1| \geq 1 - \delta\).
Proof of Second Lemma continued (If (ii) occurs, then \((v^\varepsilon_i)^* \geq 0\))

- Properties of \(u^\varepsilon, l\) \(\implies u^\varepsilon, l\) is a solution inside \(C_\varepsilon\).
- Comparison with \(v^\varepsilon_i\) and boundary continuity \(\implies u^\varepsilon, l - v^\varepsilon_i \leq C(\delta^{1/n})^\gamma\).
- Upper limit in \(\varepsilon\): \((-v^\varepsilon_i)^* \leq 0\)
- Same as \((v^\varepsilon_i)^* \geq 0\)
Choose a special $l$ such that $l$ is ARBITRARILY CLOSE to values that give (i) and values that give (ii).

**The Good Choice of $\bar{F}$**

$$\bar{F}(\phi, x_0) = \sup \left\{ l : (ii) \text{ happens for the family } (U'_{Q_1/\varepsilon})_{\varepsilon > 0} \right\}$$
Choice for $\bar{F}$

Choose a special $l$ such that $l$ is ARBITRARILY CLOSE to values that give (i) and values that give (ii).

The Good Choice of $\bar{F}$

$$\bar{F}(\phi, x_0) = \sup \left\{ l : (ii) \text{ happens for the family } (U_{Q_1/\varepsilon})_{\varepsilon>0} \right\}$$
Needed Properties for $\bar{F}$

Still need to show

**Elliptic Nonlocal Equation**

- $\bar{F}(u, x)$ is well defined whenever $u$ is bounded and "$C^{1,1}$ at the point, $x$".
- $\bar{F}(u, \cdot)$ is a continuous function in an open set, $\Omega$, whenever $u \in C^2(\Omega)$.
- Ellipticity holds: If $u$ and $v$ are $C^{1,1}$ at a point, $x$, then

$$M^- (u - v)(x) \leq \bar{F}(u, x) - \bar{F}(v, x) \leq M^+(u - v)(x).$$

**Comparison**

This follows from ellipticity and translation invariance.
Still need to show

**Elliptic Nonlocal Equation**

- \( \bar{F}(u, x) \) is well defined whenever \( u \) is bounded and “\( C^{1,1} \) at the point, \( x \)”.  
- \( \bar{F}(u, \cdot) \) is a continuous function in an open set, \( \Omega \), whenever \( u \in C^2(\Omega) \).  
- Ellipticity holds: If \( u \) and \( v \) are \( C^{1,1} \) at a point, \( x \), then  
  \[ M^- (u - v)(x) \leq \bar{F}(u, x) - \bar{F}(v, x) \leq M^+ (u - v)(x). \]

**Comparison**

This follows from ellipticity and translation invariance.
True Corrector Equation

Periodic Corrector

$\bar{F}(\phi, x_0)$ is the unique constant such that the equation,

$$F_{\phi, x_0}(w, y) = \bar{F}(\phi, x_0) \text{ in } \mathbb{R}^n$$

admits a global periodic solution, $w$. 
Corollary: Inf-Sup formula

\[ \tilde{F}(\phi, x_0) = \inf_{\{ W \text{ periodic} \}} \sup_{y \in \mathbb{R}^n} (F_{\phi, x_0}(W, y)) \]
Thank You!