Zonal Jets, Dipole EOFs, and Annular Modes

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- Characterisation of low-frequency variability (∼ 10 days +) of extratropical atmosphere important for
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- deepening understanding of atmospheric dynamics
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- Present study: how much of all of this can be understood from the kinematics of a fluctuating jet (without invoking complex dynamics)?
Empirical Orthogonal Functions (aka PCA)

- Covariance matrix of field \( u(x, t) \)

\[
C(x, x') = E \left\{ u(x, t)u(x', t) \right\} - E \left\{ u(x, t) \right\} E \left\{ u(x', t) \right\}
\]
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- EOFs: eigenfunctions of $C(x, x')$

$$\int C(x, x')E^{(j)}(x') \, dx' = \mu^{(j)}E^{(j)}(x)$$
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$$\alpha^{(j)}(t) = \int u(x, t) E^{(j)}(x) \, dx$$
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- EOFs orthogonal, PCs uncorrelated
Observed EOFs: Zonal Index and Annular Mode

Zonal Mean Geopotential $\Phi(\phi,t)$

Zonal Mean Zonal Wind $u(\phi,t)$
The Idealised Zonal Jet

Assume eddy-driven midlatitude jet described by

\[ u(x,t) = U(t) \mathcal{F} \left( \frac{x - x_c(t)}{\sigma(t)} \right) \]

where

\[ U(t) = U_0 (1 + l \xi(t)) \quad \text{jet strength} \]

\[ x_c(t) = h \lambda(t) \quad \text{jet position} \]

\[ \sigma^{-1}(t) = 1 + v \eta(t) \quad \text{jet width} \]
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- Observed: \( l << 1, h << 1, v << 1 \)
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- Strength, width correlated \( \iff \) momentum "conservation"
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- Observed: \( l << 1, \ h << 1, \ v << 1 \)

- Strength, width correlated ⇔ momentum “conservation”

- Geopotential related to zonal wind through geostrophy:
  \[ \Phi(x, t) = - \int_{x_1}^{x} f(x') u(x', t) \, dx' + \int_{x_1}^{x_2} \left( \int_{x_1}^{x} f(x') u(x') \, dx' \right) \mu(x) \, dx \]

  (where second term imposes mass conservation)
Basis Functions: Symmetric Jet

- Define normalised basis functions $F_j(x)$:

$$F_j(x) = \frac{1}{N_j} \frac{d^j \mathcal{F}}{dx^j}$$

so that

$$\int F_j^2 \, dx = 1$$
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  $F_0$ is a “monopole”, $F_1$ a “dipole”, and $F_3$ a “tripole”
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- Assume bounded jet on domain wide enough so that even/odd derivatives of $F(x)$ orthogonal:
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- By definition:

$$\frac{d}{dx} F_j(x) = \frac{N_{j+1}}{N_j} F_{j+1}(x)$$
Analytic computation of EOFs

- Exploit “smallness” of fluctuations to transform EOF integral equation into matrix equation
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- E.g.: fluctuations in position alone

\[ u'(x, t) = u(x, t) - \mathbb{E}\{u(x, t)\} = U_0 N_1 h \lambda F_1(x) + \frac{1}{2} N_2 U_0 (h \lambda)^2 F_2(x) + \ldots \]
Analytic computation of EOFs

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- E.g.: fluctuations in position alone

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and so

\[ C(x, x') = U_0^2 h^2 N_1^2 F_1(x) F_1(x') + \frac{U_0^2}{2} N_1 N_2 h^3 s_\lambda [F_1(x) F_2(x') + F_2(x) F_1(x')] \]

\[ + \frac{U_0^2}{4} N_2^2 h^4 (\kappa_\lambda + 3) F_2(x) F_2(x') + ... \]

(where \( s_\lambda, \kappa_\lambda \) skewness and kurtosis of \( \lambda \))
Writing EOF as $E_u(x) = \alpha F_1(x) + \beta F_2(x)$ gives matrix equation (to $O(h^4)$):

$$
\begin{pmatrix}
U_0^2 N_1^2 h^2 & \frac{1}{2} N_1 N_2 U_0^2 h^3 s_\lambda \\
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\end{pmatrix}
\begin{pmatrix}
\alpha \\
\beta
\end{pmatrix} = \mu
\begin{pmatrix}
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$$

If $s = 0$, "dipole" $F_1(x)$ and "tripole" $F_2(x)$ both eigenvectors.

Dipole EOF dominant unless

$N_2$ very large $s = 0$ couples these basis functions in EOFs.
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- Dipole EOF dominant unless \( \kappa_\lambda \) or \( N_2 \) very large
- \( s_\lambda \neq 0 \) couples these basis functions in EOFs
Zonal Wind: Fluctuations in Single Variables

- **Strength Alone**: only one nontrivial PCA mode
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- **Position Alone**: if \( \lambda \) unskewed

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but skewness in \( \lambda \) mixes dipole and tripole
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Width Alone: leading EOF pattern
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but skewness in \( \lambda \) mixes dipole and tripole

- **Width Alone**: leading EOF pattern

\[ E_u^{(1)}(x) = xF_1(x) \]
Zonal Wind: Strength & Position Fluctuations

For $\xi, \lambda$ independent & unskewed leading EOFs mix monopole, tripole:

\[
\begin{align*}
E^{(1)}_u(x) &= F_1(x) \\
E^{(2)}_u(x) &= \beta_0^{(+)} F_0(x) + \beta_2^{(+)} F_2(x) & \text{ mono/tripole hybrid} \\
E^{(3)}_u(x) &= \beta_0^{(-)} F_0(x) + \beta_2^{(-)} F_2(x) & \text{ mono/tripole hybrid}
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- Mixing of monopole, tripole occurs $F_0(x), F_2(x)$ not orthogonal
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Mixing of monopole, tripole occurs $F_0(x)$, $F_2(x)$ not orthogonal

Degree of mixing determined by quantities

\[
\delta = \frac{(\kappa \lambda + 2)N^2_2}{4N^2_0} \frac{h^4}{l^2}, \quad F_{02} = \int F_0(x)F_2(x) \, dx
\]
Zonal Wind: Strength & Position Fluctuations

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- Leading PC time series couple position & strength fluctuations:

  \[ \alpha_{u}^{(1)}(t) \sim (U_{0} + \xi(t)) \lambda(t) + h.o.t. \]
For Gaussian jet with $h = 0.3, l = 0.185$
Zonal Wind: Strength & Position Fluctuations

- For Gaussian jet with $h = 0.3$, $l = 0.185$

Skewness in $\lambda \Rightarrow$ EOFs asymmetric around jet axis; dipole still dominates even for strong skewness
Zonal Wind: Strength & Position Fluctuations

- Correlation of $\xi$, $\lambda$ couples dipole with other basis functions in EOFs
Zonal Wind: Strength & Position Fluctuations

- Correlation of $\xi$, $\lambda$ couples dipole with other basis functions in EOFs
- For perfect correlation $\xi = \lambda$, leading EOF mixes monopole and dipole:

$$E_u^{(1)}(x) = \frac{1}{\sqrt{1 + \epsilon^2}}(-\epsilon F_0(x) + F_1(x))$$
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For position, strength fluctuations of comparable width, coupling will be strong.
Dipole structure $F_1(x)$ will be EOF if
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- fluctuations in $\lambda$ not strongly skewed

That these fairly general facts are characteristic of the tropospheric jet in models & observations explain for dipole as generic feature of zonal wind EOFs.

While dipole arises because of position fluctuations, the associated EOF mode bundles together variability in all jet degrees of freedom.
Zonal Wind: General Case

- Dipole structure $F_1(x)$ will be EOF if
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  - position fluctuations not correlated with strength or width
Zonal Wind: General Case

- Dipole structure \( F_1(x) \) will be EOF if
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- That these fairly general facts are characteristic of the tropospheric jet in models & observations $\Rightarrow$ explanation for dipole as generic feature of zonal wind EOFs

- While dipole *arises* because of position fluctuations, the associated EOF mode bundles together variability in all jet degrees of freedom
EOFs of Dynamically Related Fields

- Geopotential related to zonal wind through linear transformation

\[ \Phi(x, t) = - \int_{x_1}^{x} f(x')u(x', t) \, dx' + \int_{x_1}^{x_2} \left( \int_{x_1}^{x} f(x')u(x') \, dx' \right) \mu(x) \, dx \]
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- Dynamically related but distinct fields will not generally have same EOFs
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- Dynamically related but distinct fields will not generally have same EOFs
- Consider \( y = Lx \); covariances related by

\[ C_{yy} = LC_{xx}L^T \]


**EOFs of Dynamically Related Fields**

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- EOF decomposition of \( x \)

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  C_{xx} = U\Lambda U^T
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  \[ C_{yy} = LC_{xx}L^T \]

- EOF decomposition of \( x \)
  \[ C_{xx} = U\Lambda U^T \]

  so

  \[ C_{yy} = (LU)\Lambda(LU)^T \]

  and EOFs of \( y \) only EOFs of \( x \) if rows of \( LU \) orthogonal
  (which they won’t be in general)
Geopotential EOFs

- Can expand $u'(x, t) = u(x, t) - \langle u(x) \rangle$ over EOF basis:

$$u'(x, t) = \sum_{j=1}^{J} \alpha_u^{(j)}(t) E_u^{(j)}(x)$$
Geopotential EOFs

- Can expand \( u'(x, t) = u(x, t) - \langle u(x) \rangle \) over EOF basis:

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u'(x, t) = \sum_{j=1}^{J} \alpha_u^{(j)}(t) E_u^{(j)}(x)\]

\[
\Rightarrow \quad \Phi'(x, t) = - \sum_{j=1}^{J} \alpha_u^{(j)}(t) \left( \int_{x_1}^{x} f(x') E_u^{(j)}(x') \, dx' - \int_{x_1}^{x_2} \int_{x_1}^{x} f(x') E_u^{j}(x') \mu(x) \, dx' \, dx \right)
\]

\[
= - \sum_{j=1}^{J} \gamma_j \alpha_u^{(j)}(t) G^{(j)}(x)
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Geopotential EOFs

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- Functions \( G_j(x) \) not generally orthogonal \(\Rightarrow\) not generally EOFs
- Mass conservation constraint influences non-orthogonality
Geopotential EOFs: Strength, Position, & Width

- Gaussian jet, neglecting sphericity of Earth ($\mu(x) = f(x) = 1$)
Geopotential: Strength & Position Fluctuations (Flat)

\[ \Phi'(x, t) \approx \alpha_u^{(1)}(t)G_1(x) + \alpha_u^{(2)}(t)G_2(x) \]
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  \[ \mathcal{I} = \int_{x_1}^{x_2} G_1(x) G_2(x) \, dx = -0.1 \]
  \[ \Rightarrow \quad E_{\Phi}^{(1)}(x) = -0.58 G_1(x) + 0.81 G_2(x) \]

- Leading EOF of \( \Phi(x, t) \) mixes EOFs of \( u(x, t) \)
  (because of mass conservation)
Annular mode structure requires both strength and position fluctuations and mixes leading two EOFs of zonal wind.
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