On the definition and properties of $p$-harmonious functions

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E. Le Gruyer; *On absolutely minimizing Lipschitz extensions and PDE $\Delta_\infty(u) = 0$*. 2007 NoDEA.

**MPR1** An asymptotic mean value property characterization of $p$-harmonic functions, 2009 preprint.

**MPR2** On the definition and properties of $p$-harmonious functions, 2009 preprint.
A directed tree with regular 3-branching $T$ consists of

- the empty set $\emptyset$,
- 3 sequences of length 1 with terms chosen from the set $\{0, 1, 2\}$,
- 9 sequences of length 2 with terms chosen from the set $\{0, 1, 2\}$,
- \ldots
- $3^r$ sequences of length $r$ with terms chosen from the set $\{0, 1, 2\}$

and so on. The elements of $T$ are called vertices.
Each vertex $v$ at level $r$ has three children (successors)

$$v_0, v_1, v_2.$$ 

Let $u: T \mapsto \mathbb{R}$ be a real valued function.

**Gradient**

The gradient of $u$ at the vertex $v$ is the vector in $\mathbb{R}^3$

$$\nabla u(v) = \{u(v_0) - u(v), u(v_1) - u(v), u(v_2) - u(v)\}.$$ 

**Divergence**

The averaging operator or *divergence* of a vector $X = (x, y, z) \in \mathbb{R}^3$ as

$$\text{div}(X) = x + y + z.$$
Harmonic Functions on Trees

Harmonic functions

A function $u$ is harmonic if it satisfies the Laplace equation

$$\text{div}(\nabla u) = 0.$$ 

The Mean Value Property

A function $u$ is harmonic if and only if it satisfies the mean value property

$$u(v) = \frac{u(v_0) + u(v_1) + u(v_2)}{3}.$$ 

Thus the values of harmonic function at level $r$ determine its values at all levels smaller than $r$. 

Juan Manfredi, Mikko Parviainen, and Julio Rossi

On the definition and properties of $p$-harmonious functions
A branch of $T$ is an infinite sequence of vertices, each followed by one of its immediate successors (this corresponds to the level $r = \infty$.) The collection of all branches forms the boundary of the tree $T$ is denoted by $\partial T$.

The mapping $g: \partial T \mapsto [0, 1]$ given by

$$g(b) = \sum_{r=1}^{\infty} \frac{b_r}{3^r}$$

(also denoted by $b$)

is a bijection (think of an expansion in base 3 of the numbers in $[0,1]$).
• We have a natural metric and natural measure in $\partial T$ inherited from the interval $[0, 1]$.
• The classical Cantor set $C$ is the subset of $\partial T$ formed by branches that don’t go through any vertex labeled 1.

The Dirichlet problem

Given a (continuous) function $f : \partial T \mapsto \mathbb{R}$ find a harmonic function $u : T \mapsto \mathbb{R}$ such that

$$\lim_{r \to \infty} u(b_r) = f(b)$$

for every branch $b = (b_r) \in \partial T$. 
Given a vertex $v \in T$ consider the subset of $\partial T$ consisting of all branches that start at $v$. This is always an interval that we denote by $I_v$.

**Solution to the Dirichlet problem, $p = 2$**

The we have

$$u(v) = \frac{1}{|I_v|} \int_{I_v} f(b) \, db.$$ 

Note that $u$ is a martingale.

We see that we can in fact solve the Dirichlet problem for $f \in L^1([0, 1])$. 

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Random Walk
Start at the top $\emptyset$. Move downward by choosing successors at random with uniform probability. When you get at $\partial T$ at the point $b$ you get paid $f(b)$ dollars.

Two player random Tug-of-War game
A coin is tossed. The player who wins the coin toss chooses the successor vertex (heads for player I, tails for player II.) The game ends when we reach $\partial T$ at a point $b$ in which case player II pays $f(b)$ dollars to player I.
Every time we run the game we get a sequence of vertices

\[ v_1, v_2, \ldots, v_k, \ldots \]

that determines a point on \( b \) the boundary \( \partial T \).
If we are at vertex \( v_1 \) and run the game, player II pays \( f(b) \) dollars to player I. Let us average out over all possible plays that start at \( v_1 \).

The value function is harmonic, \( p = 2 \).

Expected pay-off \( = \mathbb{E}^{v_1}[f(t)] = u(v_1) = \frac{1}{|l_{v_1}|} \int_{l_{v_1}} f(b) \, db \).
In this case, say that \( f \) is monotonically increasing. When player I moves he tries to move to the right. When player II moves he moves to the left. These are examples of strategies.

**Definition of Value functions**

\[
u^I(v) = \sup_{S_I} \inf_{S_{II}} \mathbb{E}^v[f(b)] \quad \text{and} \quad u^{II}(v) = \inf_{S_{II}} \sup_{S_I} \mathbb{E}^v[f(b)]
\]

**DPP (Dynamic Programming Principle)**

We have \( u^I = u^{II} \). Moreover, if we denote the common function by \( u \), it is the only function on the tree such that:

\[
u = f \text{ on } \partial T, \quad u(v) = \frac{1}{2} \left[ \max_i \{u(v_i)\} + \min_i \{u(v_i)\} \right].
\]
Let us combine random choice of successor plus tug of war. Choose $\alpha \geq 0$, $\beta \geq 0$ such that $\alpha + \beta = 1$. Start at $\emptyset$. With probability $\alpha$ the players play Tug-of-War. With probability $\beta$ move downward by choosing successors at random. When you get at $\partial T$ at the point $b$ player II pays $f(b)$ dollars to player I.

**DPP for Tug-of-War with noise, DPP = MVP**

The value function $u$ verifies the equation

$$u(v) = \frac{\alpha}{2} \left( \max_i \{ u(v_i) \} + \min_i \{ u(v_i) \} \right) + \beta \left( \frac{u(v_0) + u(v_1) + u(v_2)}{3} \right).$$
Setting
\[ \text{div}_\infty (X) = \max\{x, y, z\} + \min\{x, y, z\} \]
the value function \( u \) of the tug-of-war game satisfies
\[ \text{div}_\infty (\nabla u) = 0 \]

Setting
\[ \text{div}_p (X) = \frac{\alpha}{2} (\max\{x, y, z\} + \min\{x, y, z\}) + \beta \left( \frac{x + y + z}{3} \right) \]
the value function \( u \) of the tug-of-war game with noise satisfies
\[ \text{div}_p (\nabla u) = 0. \]

This operator is the homogeneous \( p \)-Laplacian.
The (homogeneous) $p$-Laplacian on trees

The equations

\[
\begin{align*}
\text{div}_2 (\nabla u) &= 0, \\
\text{div}_p (\nabla u) &= 0, \\
\text{div}_\infty (\nabla u) &= 0
\end{align*}
\]

DPP for Tug-of-War with noise

\[
u(v) = \frac{\alpha}{2} \left( \max_i \{u(v_i)\} + \min_i \{u(v_i)\} \right) + \beta \left( \frac{u(v_0) + u(v_1) + u(v_2)}{3} \right)
\]

1. The case $p = 2$ corresponds to $\alpha = 0$, $\beta = 1$.
2. The case $p = \infty$ corresponds to $\alpha = 1$, $\beta = 0$.
3. In general, there is no explicit solution formula for $p \neq 2$.
Suppose that \( f \) is monotonically increasing. In this case the best strategy \( S_i^* \) for player I is always to move right and the best strategy \( S_{ii}^* \) for player II always to move left. Starting at the vertex \( v \) at level \( k \)

\[
v = 0.b_1 b_2 \ldots b_k, \quad b_j \in \{0, 1, 2\}
\]

we always move either left (adding a 0) or right (adding a 1). In this case \( l_v \) is a Cantor-like set \( l_v = \{0.b_1 b_2 \ldots b_k d_1 d_2 \ldots\} \), \( d_j \in \{0, 2\} \)

**Formula for \( p = \infty \)**

\[
u(v) = \sup_{S_i} \inf_{S_{ii}} E_{S_i, S_{ii}}^v [f(b)] = E_{S_i^*, S_{ii}^*}^v [f(b)] = \int_{l_v} f(b) dC_v(b)
\]
The best strategy $S_i^*$ for player I is always to move right and the best strategy $S_{II}^*$ for player II always to move left.

**Formula for $2 \leq p \leq \infty$**

$$u(v) = \sup_{S_i} \inf_{S_{II}} \mathbb{E}^{v}_{S_i, S_{II}} [f(b)] = E_{S_i^*, S_{II}^*}^v [f(b)]$$

$$= \alpha \int_{l_v} f(b) dC_v(b) + \beta \int_{l_v} f(b) db$$
Plan of the rest of the talk:

1. Asymptotic Mean Value Properties for $p$-harmonic functions.
2. Definition, existence and uniqueness of $p$-harmonious functions.
3. Strong comparison principle for $p$-harmonious functions for $2 \leq p < \infty$.
4. Approximation of $p$-harmonic functions by $p$-harmonious functions.
1. Asymptotic mean-value properties for $p$-harmonic functions.

Let $u \in C^2(\Omega)$, $\Omega \subset \mathbb{R}^N$. Consider the Taylor expansion:

$$u(x+h) = u(x) + \langle \nabla u(x), h \rangle + \frac{1}{2} \langle D^2 u(x)h, h \rangle + o(|h|^2), \text{ as } h \to 0.$$ 

Averaging on a ball $B_\epsilon(x) \subset \Omega$ we get:

$$\int_{B_\epsilon(0)} u(x+h) \, dh = u(x) + \frac{1}{2(N+2)} \epsilon^2 \Delta(u)(x) + o(\epsilon^2), \text{ as } \epsilon \to 0.$$ 

**Lemma**

$u \in C^2(\Omega)$ is harmonic in $\Omega$ if and only if for all $x \in \Omega$

$$\int_{B_\epsilon(0)} u(x + h) \, dh = u(x) + o(\epsilon^2), \text{ as } \epsilon \to 0.$$
Since viscosity harmonic functions are harmonic in the classical sense, we indeed have:

**Lemma**

\( u \in C(\Omega) \) is harmonic in \( \Omega \) if and only if for all \( x \in \Omega \)

\[
\int_{B_\epsilon(0)} u(x + h) \, dh = u(x) + o(\epsilon^2), \quad \text{as } \epsilon \to 0
\]
The case $p = \infty$, $\nabla u(x) \neq 0$

Let $u \in C^2(\Omega)$, $\Omega \subset \mathbb{R}^N$. In the Taylor expansion, use

$$h = \epsilon \frac{\nabla u(x)}{|\nabla u(x)|} \quad \text{and} \quad h = -\epsilon \frac{\nabla u(x)}{|\nabla u(x)|},$$

add, and compute to get:

$$\frac{1}{2} \left( \sup_{B_\epsilon(x)} u + \inf_{B_\epsilon(x)} u \right) = u(x) + \epsilon^2 \Delta_\infty u(x) + o(\epsilon^2) \text{ as } \epsilon \to 0,$$

where

$$\Delta_\infty u(x) = \frac{1}{|\nabla u(x)|^2} \langle D^2 u(x) \nabla u(x), \nabla u(x) \rangle$$

is the \textit{homogeneous $\infty$-Laplacian}. 

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The case $p = \infty$, $\nabla u(x) \neq 0$

**Lemma**

$u \in C^2(\Omega)$, $\nabla u(x) \neq 0$, is $\infty$-harmonic in $\Omega$ if and only if for all $x \in \Omega$

$$
\frac{1}{2} \left( \sup_{B_\epsilon(x)} u + \inf_{B_\epsilon(x)} u \right) = u(x) + o(\epsilon^2) \text{ as } \epsilon \to 0.
$$

**Lemma**

Let $u \in C(\Omega)$ be just continuous. Suppose that for all $x \in \Omega$ we have

$$
\frac{1}{2} \left( \sup_{B_\epsilon(x)} u + \inf_{B_\epsilon(x)} u \right) = u(x) + o(\epsilon^2) \text{ as } \epsilon \to 0,
$$

then $u$ is $\infty$-harmonic in $\Omega$. 
The case $p = \infty$, $\nabla u(x) \neq 0$

The converse to the previous lemma does not hold.

**Example: Aronsson’s function near $(x, y) = (1, 0)$**

$$u(x, y) = |x|^{4/3} - |y|^{4/3}$$

Aronsson’s function is $\infty$-harmonic in the viscosity sense but it is not of class $C^2$. A calculation shows that

$$\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon^2} \left\{ \max_{B_\varepsilon(1,0)} u + \min_{B_\varepsilon(1,0)} u \right\} - u(1, 0) = \frac{1}{18}.$$  

But if an asymptotic expansion held in the classical sense, this limit would have to be zero.
The case $1 < p < \infty$, $\nabla u(x) \neq 0$

Let $u \in C^2(\Omega)$ and $\alpha, \beta$ non-negative such that $\alpha + \beta = 1$.

\[
\frac{\alpha}{2} \left( \sup_{B_\epsilon(x)} u + \inf_{B_\epsilon(x)} u \right) + \beta \int_{B_\epsilon(x)} u = u(x) \\
+ \alpha \Delta_\infty u(x) + \beta \frac{1}{(N+2)} \Delta u(x) \\
+ o(\epsilon^2), \quad \text{as } \epsilon \to 0,
\]

Let us rewrite the second order operator

\[
\alpha \Delta_\infty u(x) + \beta \frac{1}{(N+2)} \Delta u(x) = \beta \frac{1}{(N+2)} \left( \Delta u(x) + \frac{\alpha}{\beta} \frac{1}{(N+2)} \Delta_\infty u(x) \right).
\]
The case $1 < p < \infty$, $\nabla u(x) \neq 0$

Next, choose $2 < p < \infty$ such that

$$p - 2 = \frac{\alpha}{\beta \left( \frac{1}{(N+2)} \right)}.$$

We then have

$$\Delta u(x) + \frac{\alpha}{\beta \left( \frac{1}{(N+2)} \right)} \Delta_{\infty} u(x) = |\nabla u(x)|^{2-p} \text{div} \left( |\nabla u(x)|^{p-2} \nabla u(x) \right).$$

**Lemma**

$u \in C^2(\Omega)$, $\nabla u(x) \neq 0$, is $p$-harmonic in $\Omega$ if and only if for all $x \in \Omega$

$$\frac{\alpha}{2} \left( \sup_{B_{\epsilon}(x)} u + \inf_{B_{\epsilon}(x)} u \right) + \beta \int_{B_{\epsilon}(x)} u = u(x) + o(\epsilon^2), \quad \text{as} \quad \epsilon \to 0,$$
The case $1 < p < \infty$

**Lemma**

Let be $u \in C(\Omega)$. Suppose that for all $x \in \Omega$ we have

$$\frac{\alpha}{2} \left( \sup_{B_\epsilon(x)} u + \inf_{B_\epsilon(x)} u \right) + \beta \int_{B_\epsilon(x)} u = u(x) + o(\epsilon^2), \quad \text{as } \epsilon \to 0,$$

where $\alpha \geq 0$, $\beta \geq 0$, and $\alpha + \beta = 1$ and

$$\frac{p - 2}{N + 2} = \frac{\alpha}{\beta},$$

then $u$ is $p$-harmonic in $\Omega$

**Question:** Can we modify these lemmas so that they **characterize** $p$-harmonic functions?
The case $1 < p \leq \infty$

**Theorem**

$u \in C(\Omega)$ is $p$-harmonic in $\Omega$ if and only if for all $x \in \Omega$ we have that the asymptotic expansion

$$\frac{\alpha}{2} \left( \sup_{B_\epsilon(x)} u + \inf_{B_\epsilon(x)} u \right) + \beta \int_{B_\epsilon(x)} u = u(x) + o(\epsilon^2), \quad \text{as } \epsilon \to 0,$$

holds in the **VISCOSITY SENSE**, where $\alpha \geq 0$, $\beta \geq 0$, $\alpha + \beta = 1$ and

$$\frac{p - 2}{N + 2} = \frac{\alpha}{\beta}.$$

Similar results hold for $p$-subharmonic and $p$-superharmonic functions.
Asymptotic Mean Value Expansions

**Definition**

A continuous function $u$ verifies

$$u(x) = \frac{\alpha}{2} \left\{ \max_{B_{\varepsilon}(x)} u + \min_{B_{\varepsilon}(x)} u \right\} + \beta \int_{B_{\varepsilon}(x)} u(y) \, dy + o(\varepsilon^2)$$

as $\varepsilon \to 0$ *in the viscosity sense* if

(i) for every $\phi \in C^2$ that touches $u$ from below at $x$ ($u - \phi$ has a strict minimum at the point $x \in \overline{\Omega}$ and $u(x) = \phi(x)$) we have

$$\phi(x) \geq \frac{\alpha}{2} \left\{ \max_{B_{\varepsilon}(x)} \phi + \min_{B_{\varepsilon}(x)} \phi \right\} + \beta \int_{B_{\varepsilon}(x)} \phi(y) \, dy + o(\varepsilon^2).$$
Definition (continued)

(ii) for every $\phi \in C^2$ that touches $u$ from above at $x$ ($u - \phi$ has a strict maximum at the point $x \in \overline{\Omega}$ and $u(x) = \phi(x)$) we have

$$\phi(x) \leq \frac{\alpha}{2} \left\{ \max_{B_\varepsilon(x)} \phi + \min_{B_\varepsilon(x)} \phi \right\} + \beta \int_{B_\varepsilon(x)} \phi(y) \, dy + o(\varepsilon^2).$$

Sketch of the proof

$u$ $p$-harmonic $\iff$ $u$ $p$-harmonic in the viscosity sense $\iff$

Use Taylor theorem applied to the test function $\phi$.

(We can safely avoid points $x$ for which $\nabla u(x) = 0$)
2. Definition, $2 \leq p < \infty$ (\(p = \infty\) Le Gruyer)

Let \(\Omega\) be a (bounded) domain in \(\mathbb{R}^N\) and consider

\[
\Gamma_\epsilon = \{ x \in \mathbb{R}^N \setminus \Omega : \text{dist}(x, \partial \Omega) \leq \epsilon \}, \quad \Omega_\epsilon = \Omega \cup \Gamma_\epsilon
\]

The function \(u_\epsilon\) is \(p\)-harmonious in \(\Omega\) with continuous boundary values \(F : \Gamma_\epsilon \rightarrow \mathbb{R}\) if \(u_\epsilon(x) = F(x), \ x \in \Gamma_\epsilon\) and

\[
u(x) = \frac{\alpha}{2} \left\{ \sup_{B_\epsilon(x)} u_\epsilon + \inf_{\overline{B_\epsilon(x)}} u_\epsilon \right\} + \beta \int_{B_\epsilon(x)} u_\epsilon \, dy \quad \text{for every } x \in \Omega,
\]

where

\[
\alpha = \frac{p - 2}{p + N}, \quad \text{and} \quad \beta = \frac{2 + N}{p + N}.
\]

WARNING! Solutions to this equation may be discontinuous as 1-d examples show.
Fix $1 > \alpha \geq 0$, $\beta > 0$ such that $\alpha + \beta = 1$.

Fix $\varepsilon > 0$ and place a token at starting point $x_0 \in \Omega$. Move the token to the next state $x_1$ as follows:

- With probability $\alpha$ play tug-of-war: a fair coin is tossed and the winner of the toss moves the token to any $x_1 \in B_\varepsilon(x_0)$.
- With probability $\beta$ the token moves according to a uniform probability density to a random point in the ball $B_\varepsilon(x_0)$.

This procedure yields an infinite sequence of game states $x_0, x_1, \ldots$ where every $x_k$, except $x_0$, is a random variable.
A run of the game is \( \mathbf{x} = (x_0, x_1, \ldots, x_k, \ldots) \), where \( x(k) = x_k \).

The game stops the first time it hits \( \Gamma_\varepsilon \). Write

\[
\tau(\mathbf{x}) = \min\{k : x_k \in \Gamma_\varepsilon\}.
\]

The random variable \( \tau \) is a STOPPING TIME. We write

\[
\mathbf{x}(\tau(\mathbf{x})) = x_\tau.
\]

\( F : \Gamma_\varepsilon \to \mathbb{R} \) is a given (Lipschitz, bounded) payoff function. The game payoff is \( F(\mathbf{x}) = F(x_\tau) \).

Player I earns \( F(x_\tau) \) while Player II earns \( -F(x_\tau) \).
Fix strategies $S_I$ and $S_{II}$ for players I and II respectively.

Start the game at $x_0$.

The probability measure $\mathbb{P}^{x_0}_{S_I,S_{II}}$ is defined on the set of all game histories $H \subset \Omega^\infty_{\varepsilon}$ by the transition probabilities

$$
\pi_{S_I,S_{II}}(x_0, \ldots, x_k; A) = \frac{\alpha}{2} \left( \delta_{S_I}(x_0,\ldots,x_k)(A) + \delta_{S_{II}}(x_0,\ldots,x_k)(A) \right)
+ \beta \frac{|A \cap \overline{B}_\varepsilon(x_k)|}{|\overline{B}_\varepsilon(x_k)|}
$$

and Kolmogorov’s extension theorem.
Games end almost surely

\[ \mathbb{P}^{x}_{S_i,S_{II}}(H) = 1 \text{ because } \beta > 0. \]

Value of the game for player I

\[ u_{\epsilon}^{I}(x) = \sup_{S_{I}} \inf_{S_{II}} \mathbb{E}^{x}_{S_{I},S_{II}}[F(x_{\tau})] \]

Value of the game for player II

\[ u_{\epsilon}^{II}(x) = \inf_{S_{II}} \sup_{S_{I}} \mathbb{E}^{x}_{S_{I},S_{II}}[F(x_{\tau})] \]

Comparison Principle

\[ u_{\epsilon}^{I}(x) \leq u_{\epsilon}^{II}(x) \]

Juan Manfredi, Mikko Parviainen, and Julio Rossi

On the definition and properties of \( p \)-harmonious functions
THEOREM

The value functions $u^I_\varepsilon$ and $u^I_{II}$ are $p$-harmonious. They satisfy the equation

$$u(x) = \frac{\alpha}{2} \left\{ \sup_{\overline{B}_\varepsilon(x)} u + \inf_{\overline{B}_\varepsilon(x)} u \right\} + \beta \int_{\overline{B}_\varepsilon(x)} u(y) \, dy, \quad x \in \Omega,$$

$$u(x) = F(x), \quad x \in \Gamma_\varepsilon.$$

(In the case $p = \infty$ Le Gruyer showed that the mapping

$$T(u) = \frac{1}{2} \left\{ \sup_{\overline{B}_\varepsilon(x)} u + \inf_{\overline{B}_\varepsilon(x)} u \right\}$$

has a fixed point.)
Comparison I

**Theorem**

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set.

- If $v_\varepsilon$ is a $p$-harmonious function with boundary values $F_v$ in $\Gamma_\varepsilon$ such that $F_v(y) \geq u_\varepsilon^I(y)$ for $y \in \Gamma_\varepsilon$, then $v_\varepsilon(x) \geq u_\varepsilon^I(x)$ for $x \in \Omega_\varepsilon$.

- If $v_\varepsilon$ is a $p$-harmonious function with boundary values $F_v$ in $\Gamma_\varepsilon$ such that $F_v(y) \leq u_\varepsilon^I(y)$ for $y \in \Gamma_\varepsilon$, then $v_\varepsilon(x) \leq u_\varepsilon^I(x)$ for $x \in \Omega_\varepsilon$.

That is $u_\varepsilon^I$ is the smallest $p$-harmonious function with given boundary values and $u_\varepsilon^I$ is the largest $p$-harmonious function with given boundary values.
Comparison I, Proof

Player I arbitrary strategy $S_I$, player II strategy $S^0_{II}$ that almost minimizes $v_\varepsilon$,

$$v_\varepsilon(x_k) \leq \inf_{y \in \overline{B}_\varepsilon(x_{k-1})} v_\varepsilon(y) + \eta 2^{-k}$$

**Key Point**

$$M_k = v_\varepsilon(x_k) + \eta 2^{-k}$$

is a supermartingale for any $\eta > 0$.

$$\mathbb{E}^{x_0}_{S_I, S^0_{II}}[M_k \mid x_0, \ldots, x_{k-1}] = \mathbb{E}^{x_0}_{S_I, S^0_{II}}[v_\varepsilon(x_k) + \eta 2^{-k} \mid x_0, \ldots, x_{k-1}]$$

$$\leq \frac{\alpha}{2} \left\{ \inf_{y \in \overline{B}_\varepsilon(x_{k-1})} v_\varepsilon(y) + \sup_{y \in \overline{B}_\varepsilon(x_{k-1})} v_\varepsilon(y) + \eta 2^{-k} \right\}$$

$$+ \beta \int_{B_\varepsilon(x_{k-1})} v_\varepsilon \, dy + \eta 2^{-k} \leq v_\varepsilon(x_{k-1}) + \eta 2^{-(k-1)} = M_{k-1}$$
By optimal stopping

\[
\begin{align*}
  u^\varepsilon(x_0) &= \sup_{S_I} \inf_{S_{II}} \mathbb{E}_{S_I, S_{II}}^x[F(x_\tau)] \\
  &\leq \sup_{S_I} \mathbb{E}_{S_I, S_{II}}^x[F(x_\tau)] \\
  &\leq \sup_{S_I} \mathbb{E}_{S_I, S_{II}}^x[v^\varepsilon(x_\tau)] \\
  &\leq \sup_{S_I} \mathbb{E}_{S_I, S_{II}}^{x_0}[v^\varepsilon(x_\tau) + \eta 2^{-\tau}] \\
  &\leq \sup_{S_I} \mathbb{E}_{S_I, S_{II}}^{x_0}[M_\tau] \\
  &\leq \sup_{S_I} M_0 = v^\varepsilon(x_0) + \eta
\end{align*}
\]
The game has a value

**Theorem**

\[ M_k = u^\varepsilon_i(x_k) + \eta 2^{-k} \] is a supermartingale.

We have \( u^\varepsilon_i = u^\varepsilon_{II} \)

The proof is a variant of the proof of comparison. Player II follows a strategy \( S^0_{II} \) such that at \( x_{k-1} \in \Omega^\varepsilon \), he always chooses to step to a point that almost minimizes \( u^\varepsilon_i \); that is, to a point \( x_k \) such that

\[
u^\varepsilon_i(x_k) \leq \inf_{y \in \overline{B}_\varepsilon(x_{k-1})} u^\varepsilon_i(y) + \eta 2^{-k}
\]
### Theorem

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. If $u_\varepsilon$ is $p$-harmonious in $\Omega$ with a boundary data $F$, then $\sup_{\Gamma_\varepsilon} F \geq \sup_{\Omega} u_\varepsilon$. Moreover, if there is a point $x_0 \in \Omega$ such that $u_\varepsilon(x_0) = \sup_{\Gamma_\varepsilon} F$, then $u_\varepsilon$ is constant in $\Omega$.

### Theorem

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. and let $u_\varepsilon$ and $v_\varepsilon$ be $p$-harmonious with boundary data $F_u \geq F_v$ in $\Gamma_\varepsilon$. Then if there exists a point $x_0 \in \Omega$ such that $u_\varepsilon(x_0) = v_\varepsilon(x_0)$, it follows that $u_\varepsilon = v_\varepsilon$ in $\Omega$, and, moreover, the boundary values satisfy $F_u = F_v$ in $\Gamma_\varepsilon$. 

Juan Manfredi, Mikko Parviainen, and Julio Rossi

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Proof of Strong Comparison

The proof uses the fact that $p < \infty$. The strong comparison principle does not hold for $p = \infty$.

$$F_u \geq F_v \implies u_\varepsilon \geq v_\varepsilon.$$

We have

$$u_\varepsilon(x_0) = \frac{\alpha}{2} \left\{ \sup_{\overline{B}_\varepsilon(x_0)} u_\varepsilon + \inf_{\overline{B}_\varepsilon(x_0)} u_\varepsilon \right\} + \beta \int_{B_\varepsilon(x_0)} u_\varepsilon \, dy$$

and

$$v_\varepsilon(x_0) = \frac{\alpha}{2} \left\{ \sup_{\overline{B}_\varepsilon(x_0)} v_\varepsilon + \inf_{\overline{B}_\varepsilon(x_0)} v_\varepsilon \right\} + \beta \int_{B_\varepsilon(x_0)} v_\varepsilon \, dy.$$

Next we compare the right hand sides. Because $u_\varepsilon \geq v_\varepsilon$, it follows that
Proof of Strong Comparison, II

\[
\begin{align*}
\sup_{\overline{B}_\varepsilon(x_0)} u_\varepsilon - \sup_{\overline{B}_\varepsilon(x_0)} v_\varepsilon & \geq 0, \\
\inf_{\overline{B}_\varepsilon(x_0)} u_\varepsilon - \inf_{\overline{B}_\varepsilon(x_0)} v_\varepsilon & \geq 0, \quad \text{and} \\
\int_{\overline{B}_\varepsilon(x_0)} u_\varepsilon \, dy - \int_{\overline{B}_\varepsilon(x_0)} v_\varepsilon \, dy & \geq 0
\end{align*}
\]

But since

\[ u_\varepsilon(x_0) = v_\varepsilon(x_0), \]

and \( \beta > 0 \) must have \( u_\varepsilon = v_\varepsilon \) almost everywhere in \( \overline{B}_\varepsilon(x_0) \). In particular,

\[ F_u = F_v \quad \text{everywhere in } \Gamma_\varepsilon \]

since \( F_u \) and \( F_v \) are continuous. By uniqueness \( u_\varepsilon = v_\varepsilon \) everywhere in \( \Omega \).
4. Approximation of $p$-harmonic functions

Boundary Regularity Assumption

$\Omega$ bounded domain in $\mathbb{R}^n$ satisfying an exterior sphere condition: For each $y \in \partial \Omega$, there exists $B_\delta(z) \subset \mathbb{R}^n \setminus \Omega$ such that $y \in \partial B_\delta(z)$. $R > 0$ is chosen so that we always have $\Omega \subset B_{R/2}(z)$.

THEOREM

$F$ is Lipschitz in $\Gamma_\epsilon$ for small $0 < \epsilon < \epsilon_0$. Let $u$ be the unique viscosity solution to

$$
\begin{cases}
\text{div}(|\nabla u|^{p-2} \nabla u)(x) = 0, & x \in \Omega \\
u(x) = F(x), & x \in \partial \Omega,
\end{cases}
$$

and let $u_\epsilon$ be the unique $p$-harmonious function with boundary data $F$ in $\Gamma_\epsilon$, then $u_\epsilon \to u$ uniformly in $\Omega$ as $\epsilon \to 0$. 

Juan Manfredi, Mikko Parviainen, and Julio Rossi
On the definition and properties of $p$-harmonious functions
Approximation of $p$-harmonic functions, Proof I

Ascoli-Arzelá type theorem

Let $\{u_\varepsilon : u_\varepsilon : \overline{\Omega} \to \mathbb{R}, \varepsilon > 0\}$ be a set of functions such that

1. there exists $C > 0$ so that $|u_\varepsilon(x)| < C$ for every $\varepsilon > 0$ and every $x \in \overline{\Omega}$,

2. given $\eta > 0$ there are constants $r_0$ and $\varepsilon_0$ such that for every $\varepsilon < \varepsilon_0$ and any $x, x' \in \overline{\Omega}$ with $|x - x'| < r_0$ it holds

$$|u_\varepsilon(x) - u_\varepsilon(x')| < \eta.$$

Then, there exists a sequence $\varepsilon_j \to 0$ and a uniformly continuous function $u : \overline{\Omega} \to \mathbb{R}$ such that

$$u_{\varepsilon_j} \to u$$

uniformly in $\overline{\Omega}$. 
Condition 1 is clear:

\[
\min_{y \in \Gamma_\varepsilon} F(y) \leq F(x_\tau) \leq \max_{y \in \Gamma_\varepsilon} F(y) \implies \min_{y \in \Gamma_\varepsilon} F(y) \leq u_\varepsilon(x) \leq \max_{y \in \Gamma_\varepsilon} F(y).
\]

**Condition 2, OSCILLATION ESTIMATE**

The \( p \)-harmonious function \( u_\varepsilon \) with the boundary data \( F \) satisfies

\[
|u_\varepsilon(x) - u_\varepsilon(y)| \leq \text{Lip}(F)\delta + C(R/\delta)(|x - y| + o(1)),
\]

for every small enough \( \delta > 0 \) and for every two points \( x, y \in \Omega \cup \Gamma_\varepsilon \). Here \( C(R/\delta) \to \infty \) as \( R/\delta \to \infty \). Furthermore the constant in \( o(1) \) is uniform in \( x \) and \( y \).
Exterior sphere condition \( \implies \) there exists there exists \( B_\delta(z) \subset \mathbb{R}^n \setminus \Omega \) such that \( y \in \partial B_\delta(z) \).

When Player I chooses the strategy of pulling towards \( z \), denoted by \( S^z_I \), Player II an arbitrary strategy.

\[
M_k = |x_k - z| - C\varepsilon^2 k
\]

is a supermartingale for a constant \( C \) large enough independent of \( \varepsilon \).

By the optional stopping theorem

\[
\mathbb{E}^{x_0}_{S^z_I, S^z_II}[|x_\tau - z| - C\varepsilon^2 \tau] \leq |x_0 - z|
\]

\[
\mathbb{E}^{x_0}_{S^z_I, S^z_II}[|x_\tau - z|] \leq |x_0 - z| + C\varepsilon^2 \mathbb{E}^{x_0}_{S^z_I, S^z_II}[\tau]
\]
Random Walk Exit Time Estimates

Consider a random walk on $B_R(z) \setminus \overline{B}_\delta(z)$ such that when at $x_{k-1}$, the next point $x_k$ is chosen uniformly distributed in $B_\varepsilon(x_{k-1}) \cap B_R(z)$. For $\tau^* = \inf\{k : x_k \in \overline{B}_\delta(z)\}$, we have

$$\mathbb{E}^{x_0}(\tau^*) \leq \frac{C(R/\delta) \text{dist}(\partial B_\delta(z), x_0) + o(1)}{\varepsilon^2},$$

for $x_0 \in B_R(y) \setminus \overline{B}_\delta(y)$. Here $C(R/\delta) \to \infty$ as $R/\delta \to \infty$.

This is surely known by experts in probability. We proved it by showing that $g(x) = \mathbb{E}^x(\tau^*)$ can be estimated by the solution of a mixed Dirichlet-Newman problem in the ring $B_R(z) \setminus \overline{B}_\delta(z)$.