

Modeling aircraft hoses and flexible conduits

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1 Introduction

Hoses and flexible conduits are ubiquitous; from the garden hoses used to water our flower gardens to laptop power chords to the gigantic polyethylene pipes used to transport chilled water for air conditioning, we see them every day. In this paper, we are concerned with determining the equilibrium shape that a flexible conduit will take when its two ends are clamped or attached to a substrate. One of the purposes of simulation is to enable us to predict the state of a system without performing an experiment, which may be very expensive. In the context of an airplane, thousands of flexible conduits are used in the construction to encapsulate electric wires, fluids, etc. For example, see Figure 1, where hundreds of conduits are visible in the landing gear. Each of these conduits changes shape as the landing gear retracts. One can imagine the difficulty of ensuring that each tube does not tangle, kink, or break as the interacting tubes change shape.

Before simulations were available, the Boeing company would build prototype airplanes made of plywood, drill holes in the plywood, and attempt to arrange the necessary tubing in an optimal way through trial and error. Each conduit should ideally be in its low energy configuration (given its ends that are attached) to prevent any kinking or large internal stresses that can cause material damage. Furthermore, the length of the conduits should be minimized in order to use the least amount of material possible. Each extra pound used to construct the airplane translates to thousands of dollars. The simulation of the configurations of the conduits enables the Boeing company to organize and arrange the myriad of interacting flexible conduits to minimize the use of material *without* building a time-consuming, expensive plywood prototype. With the motivation of the prob-

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Figure 1: Hundreds of flexible conduits in the landing gear of an airplane.

lem, we proceed to describe our approach in modeling the shapes of the flexible conduits.

A flexible hose or conduit can be modeled as a spring. Neglecting any effects of gravity, shape memory, or torsion, a flexible hose floating in outer space with no external forces will assume a straight shape with no curvature— analogous to the equilibrium position of a spring. When a flexible conduit is bent into a non-straight shape, an internal stress is induced by the compression and stretching of the material below and above the center line of the bending. In regions of high curvature, the material is under a high amount of stress because the material must be compressed and stretched a lot above and below the center line of the conduit in order to assume its shape. An expression can be derived by modeling the compression and stretching of the conduit with Hookes law that gives the potential energy due to bending of a flexible conduit as proportional to

$$J = \int_{s_0}^{s_1} \kappa^2(s) ds. \quad (1.1)$$

We use this expression for the potential energy of a bent conduit to find the equilibrium shape when the ends of the conduit are clamped at certain positions and angles. In using the above formulation for potential energy, we neglect gravitational forces under the assumption that the rigidity of the conduit is large in comparison to its own mass. Also, torsional effects and shape memory are neglected under the assumption that the hoses will be physically attached to their substrates with a swivel mechanism. Under these assumptions, a flexible conduit constrained at the boundaries will assume the shape that minimizes the integral of the square of its curvature.

Once the conditions under which the ends of the conduits are configured are established, along with any conduit length constraints, our mathematical approach to minimizing the potential energy is the calculus of variations. The functional in equation (1.1) takes a curve, represented by function(s) of some sort, and assigns a single value to it— the potential energy of the curve. The calculus of variations is an extension of calculus that treats the problem of minimizing an integral over a space

of functions. In this paper, we use different approaches to represent the shape of a conduit. Then, we use the calculus of variations and the Lagrangian multiplier method to determine the equations that describe the shape that the conduit will take in its resting state under certain constraints along with a sufficient condition to check if the configuration is a local minimum. We show the result of the numerical computations of the resting states, followed by a discussion of the advantages of each approach. In the last section, we present the numerical methods used to solve the equations obtained by the calculus of variations, namely finite element collocation and the three-stage Lobatto IIIa formula (a built-in method in MATLAB).

2 Preliminaries

A curve in a d -dimensional space ($d = 2, 3$) can be characterized by a function $\mathbf{y}(t) : [t_0, t_1] \rightarrow \mathbb{R}^d$. The curvature of the curve κ at a point t is defined to be $\kappa(t) = 1/R(t)$, where $R(t)$ is the radius of the circle which is tangent to the curve at point t . It is easy to show that

$$\kappa(t) = \frac{\|\mathbf{y}' \times \mathbf{y}''\|}{\|\mathbf{y}'\|^3}.$$

If we pick $t = s$ such that $\|\mathbf{y}'(s)\|$ is a constant, then $\kappa(s)$ is proportional to $\|\mathbf{y}''(s)\|$.

Our objective is to minimize J in (1.1) among all functions \mathbf{y} which satisfied proper boundary conditions. We introduce the calculus of variations to find the minimizer.

Suppose we have the following minimization problem

$$\text{Minimize } J(\mathbf{y}) = \int_{t_0}^{t_1} f(t, \mathbf{y}(t), \mathbf{y}'(t), \mathbf{y}''(t)) dt. \quad (2.1)$$

Theorem 2.1 (See [1, 2]). *The necessary condition for \mathbf{y} being a minimizer of J is that $\frac{d}{d\epsilon} J(\mathbf{y} + \epsilon \mathbf{v})|_{\epsilon=0} = 0$ for all $\mathbf{v} \in C_c^\infty([t_0, t_1])^d$.*

For functional J in (2.1), we write

$$\begin{aligned} 0 = \frac{d}{d\epsilon} J(\mathbf{y} + \epsilon \mathbf{v})|_{\epsilon=0} &= \int_{t_0}^{t_1} \frac{d}{d\epsilon} f(t, \mathbf{y} + \epsilon \mathbf{v}, \mathbf{y}' + \epsilon \mathbf{v}', \mathbf{y}'' + \epsilon \mathbf{v}'')|_{\epsilon=0} dt \\ &= \int_{t_0}^{t_1} (f_{\mathbf{y}} \cdot \mathbf{v} + f_{\mathbf{y}'} \cdot \mathbf{v}' + f_{\mathbf{y}''} \cdot \mathbf{v}'') dt \\ &= \int_{t_0}^{t_1} \left(f_{\mathbf{y}} \cdot \mathbf{v} - \frac{d}{dt} f_{\mathbf{y}'} \cdot \mathbf{v} + \frac{d^2}{dt^2} f_{\mathbf{y}''} \cdot \mathbf{v} \right) dt \\ &= \int_{t_0}^{t_1} \left(f_{\mathbf{y}} - \frac{d}{dt} f_{\mathbf{y}'} + \frac{d^2}{dt^2} f_{\mathbf{y}''} \right) \cdot \mathbf{v} dt \end{aligned}$$

for all $v \in C_c^\infty([t_0, t_1])^d$. Hence we obtain *Euler-Lagrange* differential equations for the minimization problem:

$$f_y - \frac{d}{dt} f_{y'} + \frac{d^2}{dt^2} f_{y''} = 0 \quad (2.2)$$

Theorem 2.2. *The sufficient condition for the solution y of (2.2) being a minimizer of J is that $\frac{d^2}{d\epsilon^2} J(y + \epsilon v)|_{\epsilon=0} > 0$ for all $v \in C_c^\infty([t_0, t_1])^d$.*

Now we state an important lemma to help estimate the sufficient condition in Section 3 and 4.

Lemma 2.3 (Wirtinger's inequality). *Suppose v is a C^1 function such that $v(0) = v(a) = 0$. Then*

$$\int_0^a |v|^2 \leq \left(\frac{a}{\pi}\right)^2 \int_0^a |v'|^2.$$

3 Non parametric approach

3.1 Two dimensional case without length constraint

In \mathbb{R}^2 space with coordinate (x, y) , $y = y(x)$ represents the equation of a plane curve. Let $y : [0, a] \rightarrow \mathbb{R}$ be a C^4 function. Then we have

$$\begin{aligned} \text{Curvature } \kappa &= \frac{y''}{1 + (y')^2}. \\ \text{Energy } J &= \int_0^a \left(\frac{y''}{1 + (y')^2} \right)^2 \sqrt{1 + (y')^2} dx. \end{aligned}$$

Now our goal under this setting is

$$\text{Minimize } J(y) = \int_0^a \frac{(y'')^2}{(1 + (y')^2)^{\frac{5}{2}}} dx. \quad (3.1)$$

Here $f(x, y, y', y'') = \frac{(y'')^2}{(1 + (y')^2)^{\frac{5}{2}}}$. From (2.2) we have the corresponding Euler-Lagrange equation of (3.1):

$$5(-1 + 6(y')^2)(y'')^3 - 20(y' + (y')^3)y''y^{(3)} + 2(1 + (y')^2)^2 y^{(4)} = 0. \quad (3.2)$$

Theorem 3.1. *If $f_{y''y''} > 0$ for all $x \in [0, a]$, then the sufficient condition for the solution of (3.2) being a minimizer of $J(y)$ in (3.1) is*

$$f_{y'y'} - \frac{d}{dx} f_{y'y''} + \frac{\pi^2}{a^2} \min_x f_{y''y''} > 0 \text{ for all } x \in [0, a].$$

Proof. Since $f_{y''y''} > 0$, we have

$$\begin{aligned}
\frac{d^2}{d\epsilon^2} J(y + \epsilon v)|_{\epsilon=0} &= \int_0^a (v')^2 f_{y'y'} + 2v'v'' f_{y'y''} + (v'')^2 f_{y''y''} dx \\
&= \int_0^a (v')^2 (f_{y'y'} - \frac{d}{dx} f_{y'y''}) + (v'')^2 f_{y''y''} dx \\
&\geq \int_0^a (v')^2 (f_{y'y'} - \frac{d}{dx} f_{y'y''}) dx + \min_x f_{y''y''} \int_0^a (v'')^2 dx \\
&\geq \int_0^a (v')^2 (f_{y'y'} - \frac{d}{dx} f_{y'y''}) dx + \frac{\pi^2 \min_x f_{y''y''}}{a^2} \int_0^a (v')^2 dx \\
&= \int_0^a (v')^2 \left(f_{y'y'} - \frac{d}{dx} f_{y'y''} + \frac{\pi^2 \min_x f_{y''y''}}{a^2} \right) dx,
\end{aligned}$$

where we applied Lemma 2.3 to the second inequality. ■

Note that $f_{y''y''} = \frac{2}{(1 + (y')^2)^{\frac{5}{2}}} > 0$ and

$$\begin{aligned}
&f_{y'y'} - \frac{d}{dx} f_{y'y''} + \frac{\pi^2}{a^2} \min f_{y''y''} \\
&= \frac{5[(y'')^2 + 2y'(1 + (y')^2)y^{(3)} - 6(y'')^2(y')^2]}{(1 + (y')^2)^{\frac{9}{2}}} + \frac{\pi^2}{a^2} \min \frac{2}{(1 + (y')^2)^{\frac{5}{2}}}.
\end{aligned}$$

Therefore, if a solution y of (3.2) satisfies

$$S(x) := \frac{5[(y'')^2 + 2y'(1 + (y')^2)y^{(3)} - 6(y'')^2(y')^2]}{(1 + (y')^2)^{\frac{9}{2}}} + \frac{\pi^2}{a^2} \min \frac{2}{(1 + (y')^2)^{\frac{5}{2}}} > 0, \quad (3.3)$$

it is a minimizer of J .

3.2 Two dimensional case with length constraint

Next, we consider the case when the total length L of the curve y is fixed, that is

$$\int_0^a \sqrt{1 + (y')^2} dx = L. \quad (3.4)$$

and we minimize energy $J(y)$ such that (3.4) holds. Applying the Lagrangian multiplier method, we modify (3.1) to be the following:

$$\text{Minimize } \tilde{J}_\lambda(y) = \int_0^a \frac{(y'')^2}{(1 + (y')^2)^{\frac{5}{2}}} dx + \lambda \left(\int_0^a \sqrt{1 + (y')^2} dx - L \right). \quad (3.5)$$

Let

$$f(x, y, y', y'') := \frac{(y'')^2}{(1 + (y')^2)^{\frac{5}{2}}} + \lambda \left(\sqrt{1 + (y')^2} - \frac{L}{a} \right).$$

From (2.2) we have the corresponding *Euler-Lagrange* equation:

$$5(-1+6(y')^2)(y'')^3 - 20(y' + (y')^3)y''y^{(3)} + 2(1+(y')^2)^2y^{(4)} - \lambda y''(1+(y')^2)^3 = 0. \quad (3.6)$$

Suppose y_λ is a solution of (3.6) for a fixed λ . Define

$$\Upsilon(\cdot) : \quad \lambda \mapsto \int_0^a \sqrt{1 + (y'_\lambda)^2} dx.$$

If there exists a λ^* such that $\Upsilon(\lambda^*) = L$, then $y^* := y_{\lambda^*}$ is a solution of the length constraint problem.

In addition, if λ^* and y^* satisfy

$$\frac{5[(y'')^2 + 2y'(1 + (y')^2)y^{(3)} - 6(y'')^2(y')^2] + \lambda(1 + (y')^2)^3}{(1 + (y')^2)^{\frac{9}{2}}} + \frac{\pi^2}{a^2} \min_x \frac{2}{(1 + (y')^2)^{\frac{5}{2}}} > 0,$$

then y^* is a minimizer of the length constraint problem.

3.3 Three dimensional case without length constraint

Without loss of generality, we represent the curve (x, y, z) in 3-D space using one of its components, z , i.e.

$$(x, y, z) = (x(z), y(z), z), \quad z \in [a, b].$$

Again, the energy function, which is the integral of curvature squared can be written as

$$J(x, y) = \int_a^b f(x', x'', y', y'') dz$$

where

$$f(x', x'', y', y'') = \frac{x''^2 + y''^2 + (x'y'' - x''y')^2}{(1 + x'^2 + y'^2)^{\frac{5}{2}}} \quad (3.7)$$

The corresponding *Euler-Lagrange* equations become

$$\begin{cases} A_1 \cdot x^{(4)} + B_1 \cdot y^{(4)} + R_1 = 0 \\ A_2 \cdot x^{(4)} + B_2 \cdot y^{(4)} + R_2 = 0, \end{cases} \quad (3.8)$$

where

$$\begin{aligned}
A_1 &:= 2(1 + x'^2 + y'^2)^2(1 + y'^2), \\
B_1 &:= -2x'y'(1 + x'^2 + y'^2)^2, \\
R_1 &:= (1 + x'^2 + y'^2) \{ 2(1 + x'^2 + y'^2)(2y^{(3)}(x''y' - x'y'') + 3y''(x^{(3)}y' \\
&\quad + x'y^{(3)}) + y'(x^{(3)}y'' - x''y^{(3)})) - 2(x'x'' + y'y'')(3x^{(3)} + y''(x''y' \\
&\quad - x'y'') + 3y'(x^{(3)}y' - x'y^{(3)})) - 10(x'x^{(3)} + x''^2 + y'y^{(3)} + y''^2) \\
&\quad \cdot (x'' + y'(x''y' - x'y'')) + 5[x''^3 + x''y''^2 + x''(x'y'' - x''y')^2 \\
&\quad + 2x'(x''x^{(3)} + y''y^{(3)} - (x'y'' - x''y')(x^{(3)}y' - x'y^{(3)}))] \} \\
&\quad - 7(x'x'' + y'y'')[2(1 + x'^2 + y'^2)(x^{(3)} + 2y''(x''y' - x'y'') \\
&\quad + y'(x^{(3)}y' - x'y^{(3)})) - 10(x'x'' + y'y'')(x'' + y'(x''y' - x'y'')) \\
&\quad + 5x'(x''^2 + y''^2 + (x'y'' - x''y')^2)],
\end{aligned}$$

$$\begin{aligned}
A_2 &:= -2x'y'(1 + x'^2 + y'^2)^2, \\
B_2 &:= 2(1 + x'^2 + y'^2)^2(1 + x'^2), \\
R_2 &:= (1 + x'^2 + y'^2) \{ 2(1 + x'^2 + y'^2)(-2x^{(3)}(x''y' - x'y'') + 3x''(x^{(3)}y' \\
&\quad + x'y^{(3)}) - x'(x^{(3)}y'' - x''y^{(3)})) - 2(x'x'' + y'y'')(3y^{(3)} - x''(x''y' \\
&\quad - x'y'') - 3x'(x^{(3)}y' - x'y^{(3)})) - 10(x'x^{(3)} + x''^2 + y'y^{(3)} + y''^2) \\
&\quad \cdot (y'' - x'(x''y' - x'y'')) + 5[y''^3 + y''x''^2 + y''(x'y'' - x''y')^2 \\
&\quad + 2y'(x''x^{(3)} + y''y^{(3)} - (x'y'' - x''y')(x^{(3)}y' - x'y^{(3)}))] \} \\
&\quad - 7(x'x'' + y'y'')[2(1 + x'^2 + y'^2)(y^{(3)} - 2x''(x''y' - x'y'') \\
&\quad - x'(x^{(3)}y' - x'y^{(3)})) - 10(x'x'' + y'y'')(y'' - x'(x''y' - x'y'')) \\
&\quad + 5y'(x''^2 + y''^2 + (x'y'' - x''y')^2)].
\end{aligned}$$

3.4 Three dimensional case with length constraint

The constrained energy in this case is

$$\text{Minimize } \tilde{J}_\lambda(y) = \int_a^b f(x', x'', y', y'') dx + \lambda \left(\int_a^b \sqrt{1 + (x')^2 + (y')^2} dx - L \right). \quad (3.9)$$

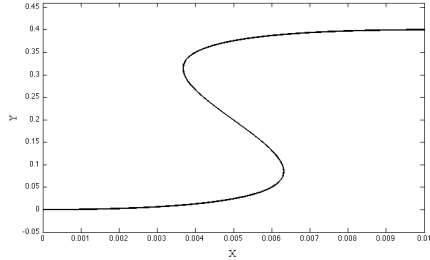
where f is defined in (3.7). The resulting *Euler-Lagrange* equations are

$$\begin{cases} \frac{A_1 \cdot x^{(4)} + B_1 \cdot y^{(4)} + R_1}{(1 + (x')^2 + (y')^2)^{9/2}} - \lambda \frac{x'' + x''(y')^2}{(1 + (x')^2 + (y')^2)^{3/2}} = 0 \\ \frac{A_2 \cdot x^{(4)} + B_2 \cdot y^{(4)} + R_2}{(1 + (x')^2 + (y')^2)^{9/2}} - \lambda \frac{y'' + y''(x')^2}{(1 + (x')^2 + (y')^2)^{3/2}} = 0, \end{cases} \quad (3.10)$$

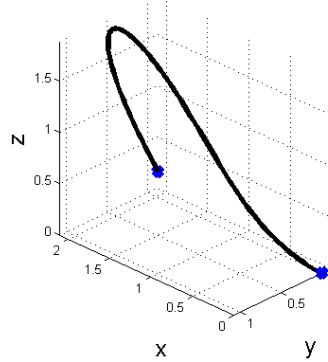
where A, B, C, D is same as in (3.8).

3.5 Non-parametric approach is limited

The representation of a curve used in Section 3.1-3.4 is a natural approach. However, if the the desired curve is not a function of the independent variable (see Figure 2), this approach could not provide right solution. A parametric approach can successfully resolve this issue and will be introduced in section 4.



This curve is a minimizer of J in (3.1) with boundary conditions: $y(0) = 0, y'(0) = 0.1$, $y(0.01) = 1, y'(0.01) = 0.1$. It cannot be represented by $(x, y(x))$.



This curve is a minimizer of J with fixed length $L = 5$. One endpoint is $(0,0,0)$ with velocity $((3,1,1))$. The other end is $(1,1,1)$ with velocity $(-1,-1,-3)$. It cannot be represented by $(x(z), y(z), z)$.

Figure 2: Examples that non-parametric approach fails

4 Parametric Approach

To overcome the drawback mentioned in Section 3.5, we parametrize the curve with parameter s and describe the flexible conduit with $\mathbf{y} \in C^4([0, 1], \mathbb{R}^3)$. Further, we choose the parametrization speed $\|\mathbf{y}'(s)\|$ to be constant so that the complicated curvature expression κ reduces to the simple expression $\|\mathbf{y}'(s)\|$. In this scenario, the total potential energy is proportional to

$$J(\mathbf{y}) = \int_0^1 \mathbf{y}'' \cdot \mathbf{y}'' ds. \quad (4.1)$$

4.1 Parametric approach without length constraint

First, we consider the case when the total length of a curve $\mathbf{y} = \mathbf{y}(s)$ is unrestricted. By differentiating $\|\mathbf{y}'(s)\| = \text{constant}$, we have $\mathbf{y}'(s) \cdot \mathbf{y}''(s) = 0$ for all $s \in [0, 1]$. Now applying the Lagrangian multiplier method, we obtain

$$\text{Minimize } J_\lambda(\mathbf{y}) = \int_0^1 (\mathbf{y}'' \cdot \mathbf{y}'' + \lambda (\mathbf{y}' \cdot \mathbf{y}'')^2) ds. \quad (4.2)$$

Note that $\int_0^1 (\mathbf{y}' \cdot \mathbf{y}'')^2 ds = 0$ implies $\mathbf{y}'(s) \cdot \mathbf{y}''(s) = 0$ for all $s \in [0, 1]$. Let

$f(s, \mathbf{y}, \mathbf{y}', \mathbf{y}'') = \mathbf{y}'' \cdot \mathbf{y}'' + \lambda (\mathbf{y}' \cdot \mathbf{y}'')^2$. From (3.2) the corresponding *Euler-Lagrange* equations are

$$\begin{aligned} 0 &= -\frac{d}{ds} f_{\mathbf{y}'} + \frac{d^2}{ds^2} f_{\mathbf{y}''} \\ &= \mathbf{y}^{(4)} + \lambda(\mathbf{y}'' \cdot \mathbf{y}'' + \mathbf{y}' \cdot \mathbf{y}^{(3)})\mathbf{y}'' + \lambda(3\mathbf{y}'' \cdot \mathbf{y}^{(3)} + \mathbf{y}' \cdot \mathbf{y}^{(4)})\mathbf{y}'. \end{aligned}$$

4.2 Parametric approach with length constraint

Next, we consider the case when the total length L of a curve \mathbf{y} is fixed and constant speed, that is

$$\int_0^1 \|\mathbf{y}'\| ds = L. \quad (4.3)$$

We minimize energy $J(\mathbf{y})$ such that (4.3) holds. Now applying the Lagrangian multiplier method we obtain

$$\text{Minimize } J_\lambda(\mathbf{y}(s)) = \int_0^1 \{ \mathbf{y}'' \cdot \mathbf{y}'' + \lambda (\mathbf{y}' \cdot \mathbf{y}' - L^2)^2 \} ds. \quad (4.4)$$

Let $f(\mathbf{y}', \mathbf{y}'') = \mathbf{y}'' \cdot \mathbf{y}'' + \lambda(\mathbf{y}' \cdot \mathbf{y}' - L^2)^2$. From (3.2) the corresponding *Euler-Lagrange* equations are

$$\begin{aligned} 0 &= -\frac{d}{ds} f_{\mathbf{y}'} + \frac{d^2}{ds^2} f_{\mathbf{y}''} \\ &= \mathbf{y}^{(4)} - 2\lambda(\mathbf{y}' \cdot \mathbf{y}' - L^2)\mathbf{y}'' - 4\lambda(3\mathbf{y}' \cdot \mathbf{y}'')\mathbf{y}'. \end{aligned} \quad (4.5)$$

Theorem 4.1. *Let \mathbf{y}_* be the solution of (4.5) when $\lambda = \lambda_*$. Then the sufficient condition for $\mathbf{y}_* = (y_1, y_2, y_3)$ being a minimizer of (4.1) with length constraint (4.3) is*

1. $\pi^2 + 2\lambda(\mathbf{y}'_* \cdot \mathbf{y}'_* - L^2) > 0$, if $\lambda_* > 0$
2. $\pi^2 + 8\lambda c^* + 2\lambda(\mathbf{y}'_* \cdot \mathbf{y}'_* - L^2) > 0$, if $\lambda_* < 0$

where $c^* = \max\{(1 + c_1 + \frac{1}{c_3})(y'_1)^2, (1 + \frac{1}{c_1} + c_2)(y'_2)^2, (1 + \frac{1}{c_2} + c_3)(y'_3)^2\}$, $c_i > 0, i = 1, 2, 3$.

Proof. 1. Since $\lambda > 0$, we obtain

$$\begin{aligned}
\frac{d^2}{d\epsilon^2} J(\mathbf{y} + \epsilon \mathbf{v})|_{\epsilon=0} &= \int_0^1 2(\mathbf{v}'' \cdot \mathbf{v}'') + 16\lambda(\mathbf{y}' \cdot \mathbf{v}')^2 + 4\lambda(\mathbf{v}' \cdot \mathbf{v}')(\mathbf{y}' \cdot \mathbf{y}' - L^2) ds \\
&\geq \int_0^1 2(\mathbf{v}'' \cdot \mathbf{v}'') + 4\lambda(\mathbf{v}' \cdot \mathbf{v}')(\mathbf{y}' \cdot \mathbf{y}' - L^2) ds \\
&\geq \int_0^1 2\pi^2(\mathbf{v}' \cdot \mathbf{v}') + 4\lambda(\mathbf{v}' \cdot \mathbf{v}')(\mathbf{y}' \cdot \mathbf{y}' - L^2) ds \\
&= \int_0^1 2(\mathbf{v}' \cdot \mathbf{v}')[\pi^2 + 2\lambda(\mathbf{y}' \cdot \mathbf{y}' - L^2)] ds, \\
&> 0,
\end{aligned}$$

where we apply Lemma 2.3 to the second inequality.

2. Since $\lambda < 0$, we obtain

$$\begin{aligned}
\frac{d^2}{d\epsilon^2} J(\mathbf{y} + \epsilon \mathbf{v})|_{\epsilon=0} &= \int_0^1 2(\mathbf{v}'' \cdot \mathbf{v}'') + 16\lambda(\mathbf{y}' \cdot \mathbf{v}')^2 + 4\lambda(\mathbf{v}' \cdot \mathbf{v}')(\mathbf{y}' \cdot \mathbf{y}' - L^2) ds \\
&\geq \int_0^1 2(\mathbf{v}'' \cdot \mathbf{v}'') + 16\lambda c^*(\mathbf{v}' \cdot \mathbf{v}') + 4\lambda(\mathbf{v}' \cdot \mathbf{v}')(\mathbf{y}' \cdot \mathbf{y}' - L^2) ds \\
&\geq \int_0^1 2\pi^2(\mathbf{v}' \cdot \mathbf{v}') + 16\lambda c^*(\mathbf{v}' \cdot \mathbf{v}') + 4\lambda(\mathbf{v}' \cdot \mathbf{v}')(\mathbf{y}' \cdot \mathbf{y}' - L^2) ds \\
&= \int_0^1 2(\mathbf{v}' \cdot \mathbf{v}')[\pi^2 + 8\lambda c^* + 2\lambda(\mathbf{v}' \cdot \mathbf{v}')(\mathbf{y}' \cdot \mathbf{y}' - L^2)] ds \\
&> 0,
\end{aligned}$$

where for the first inequality, we use the fact that

$$(\mathbf{y}' \cdot \mathbf{v}')^2 \leq c^*(\mathbf{v}' \cdot \mathbf{v}'). \quad (4.6)$$

And we apply Lemma 2.3 for the second inequality. The proof of (4.6) is the following:

$$\begin{aligned}
(\mathbf{y}' \cdot \mathbf{v}')^2 &= (y'_1 v'_1 + y'_2 v'_2 + y'_3 v'_3)^2 \\
&= (y'_1 v'_1)^2 + (y'_2 v'_2)^2 + (y'_3 v'_3)^2 + 2y'_1 v'_1 y'_2 v'_2 + 2y'_2 v'_2 y'_3 v'_3 + 2y'_3 v'_3 y'_1 v'_1 \\
&\leq (y'_1 v'_1)^2 + (y'_2 v'_2)^2 + (y'_3 v'_3)^2 + c_1 (y'_1 v'_1)^2 \\
&\quad + \frac{1}{c_1} (y'_2 v'_2)^2 + c_2 (y'_2 v'_2)^2 + \frac{1}{c_2} (y'_3 v'_3)^2 + c_3 (y'_3 v'_3)^2 + \frac{1}{c_3} (y'_1 v'_1)^2 \\
&= (1 + c_1 + \frac{1}{c_3})(y'_1 v'_1)^2 + (1 + c_2 + \frac{1}{c_1})(y'_2 v'_2)^2 + (1 + c_3 + \frac{1}{c_2})(y'_3 v'_3)^2 \\
&\leq c^*((v'_1)^2 + (v'_2)^2 + (v'_3)^2) \\
&= c^*(\mathbf{v}' \cdot \mathbf{v}').
\end{aligned}$$

■

5 Numerical Results

In this section, we explain numerical schemes used to solve the *Euler-Lagrange* equations, namely the *three-stage Lobatto IIIa formula* [4] for non-parametric approach, and *finite element collocation method* for parametric approach. We also provide numerical results of minimum energy curves in two and three dimensional spaces.

5.1 Non-parametric approach

Example 1. We consider a two dimensional boundary value problem without length constraint. The boundary condition is given by

$$y(0) = 0, \quad y'(0) = -0.077, \quad y(0.2) = 0, \quad y'(0.2) = -0.077. \quad (5.1)$$

The fourth order ordinary differential equation (3.2) can be converted to a system of first order equations

$$\frac{d}{dx} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} y_2 \\ y_3 \\ y_4 \\ \left(-5(-1 + 6y_2^2)y_3^3 + 20(y_2 + y_2^3)y_3y_4^3 \right) / (2(1 + y_2^2)^2) \end{pmatrix} \quad (5.2)$$

where $y_1 := y, y_2 := y', y_3 := y'', y_4 := y'''$.

In Matlab, we apply `bvp4c` function to solve (5.1)-(5.2). Figure 3 shows the solution of the problem. We also check that the solution satisfies the sufficient condition (3.3). Therefore, the solution is a minimizer of the energy J .

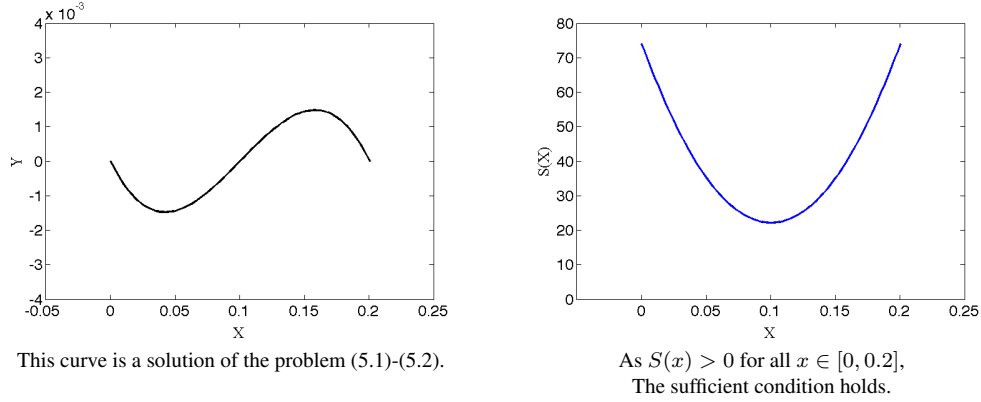


Figure 3: Results for Example 1

Example 2. Consider the boundary value problem in two dimensional space with fixed length $L = 1.023$. The boundary conditions are given by

$$y(0) = 0, \quad y'(0) = 0.3, \quad y(1) = 0, \quad y'(1) = 0.1. \quad (5.3)$$

There exist $\lambda_1^* = -54.72$ and $\lambda_2^* = -200$ such that $\Upsilon(\lambda_1^*) = \Upsilon(\lambda_2^*) = L$. Hence, there are two solutions $y_{\lambda_1^*}$ and $y_{\lambda_2^*}$ for (3.1) with constraint (3.4) (see Figure 4). Sufficient conditions should be checked to determine a minimizer.

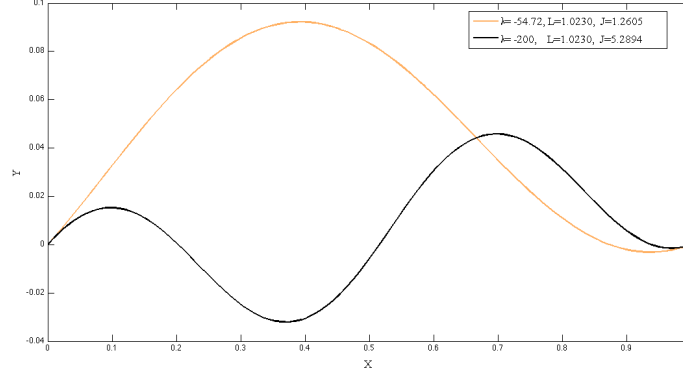


Figure 4: Two solutions $y_{\lambda_1^*}$ and $y_{\lambda_2^*}$ with the same length

Example 3 We consider the boundary value problem in three dimensional space with length constraint $L = 1.8541$. The boundary conditions are given by

$$x(0) = y(0) = 0, x'(0) = y'(0) = 1, x(1) = 0, x'(1) = -1, y(1) = 1, y'(1) = -1. \quad (5.4)$$

Figure 5 shows the solution of (3.9) (5.4).

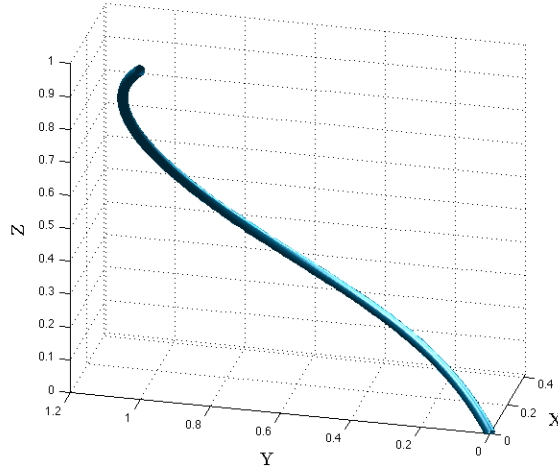


Figure 5: Solution of Example 3

5.2 Parametric approach using finite element collocation method

In solving the parameterized variational problem in two dimensions, we consider $\mathbf{y}(s) = (x(s), y(s))$ on $[0, 1]$. The *Euler-Lagrange* equation is a system of fourth

order ODEs in s :

$$\begin{aligned} x^{(4)}(s) &= f_x(x'(s), x''(s), y'(s), y''(s)) \text{ on } s \in [0, 1] \\ y^{(4)}(s) &= f_y(x'(s), x''(s), y'(s), y''(s)), \end{aligned} \quad (5.5)$$

subject to the boundary conditions

$$\begin{aligned} g_n(x(0), x'(0), y(0), y'(0)) &= 0 \text{ for } n = 1, 2, 3, 4 \\ h_n(x(1), x'(1), y(1), y'(1)) &= 0 \text{ for } n = 1, 2, 3, 4. \end{aligned} \quad (5.6)$$

In this section, we present a finite element method, collocation, to approximate the solution to the above system. The generalization to higher dimensions and more complicated boundary conditions is trivial.

The finite element method is to approximate $x(s)$ and $y(s)$ as a finite sum of basis functions that are chosen to represent the space of functions to which $x(s)$ and $y(s)$ belong:

$$\begin{aligned} \hat{x}(s) &:= \sum_{i=-2}^{N+2} \alpha_i \phi_i(s) \\ \hat{y}(s) &:= \sum_{i=-2}^{N+2} \beta_i \phi_i(s). \end{aligned} \quad (5.7)$$

Once an appropriate set of basis functions $\{\phi_i(s)\}$ is chosen, finding the solution to (5.5) amounts to finding the coefficients α_i and β_i . The easiest choices for the basis functions ϕ_i for a fourth order ODE are the quintic B-spline polynomials, one of which is plotted in Figure 6. A polynomial of at least degree 5 is needed in order to represent the fourth derivative in the ODE in system (5.5).

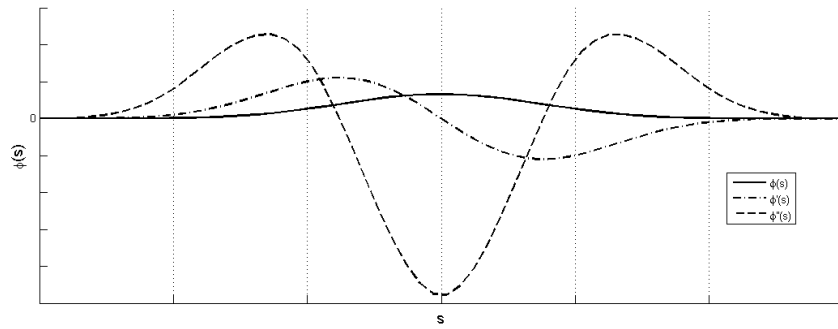


Figure 6: A quintic B-spline polynomial and its first and second derivatives. Elements are separated by the vertical dashed lines. The peak of the function $\phi_i(s)$ is at s_i .

To define the basis functions, we first discretize the s interval $[0, 1]$ into $N + 1$ points $s_i := ih$ with uniform spacing h , as in Figure 7. Two ghost points are

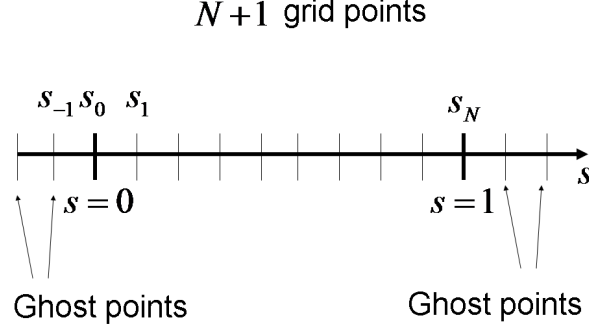


Figure 7: The domain $[0, 1]$ is discretized into $N + 1$ grid points with two ghost points.

introduced at the end points ($i = -2, -1, N + 1, N + 2$) for a proper number of degrees of freedom.

The quintic B-spline $\phi_i(s)$ has the following properties:

- ϕ_i is nonzero only inside the interval $[s_{i-3}, s_{i+3}]$.
- ϕ_i is continuous on $[0, 1]$ and so are its first, second, third, and fourth derivatives.

The continuity of the first, second, and third derivative is necessary for representing a sufficiently smooth function $x(s)$ and $y(s)$ in the system (5.5). The continuity of the fourth derivatives restricts us to the situation where f_x and f_y are continuous with respect to all of their components, and this is the case for modeling flexible conduits. Inside of each element $[s_i, s_{i+1}]$, a weighted sum with respect to α_i (β_i) of six different basis functions represents the value of $x(s)$ ($y(s)$). The forms of the quintic B-spline polynomials can be found in [3].

For the approximation in equations (5.7), we have a total of $2N + 10$ degrees of freedom. Forcing the boundary conditions in equations (5.6) to hold provides 8 constraints on $\{\alpha_i, \beta_i\}$. The method of collocation is to provide the $2N + 2$ constraints to fully specify the system by forcing $x(s)$ and $y(s)$ to satisfy the ODEs in system (5.5) at the grid points s_i in $[0, 1]$. That is:

$$\begin{aligned} \hat{x}^{(4)}(s_i) &= f_x(\hat{x}'(s_i), \hat{x}''(s_i), \hat{y}'(s_i), \hat{y}''(s_i)) \\ \hat{y}^{(4)}(s_i) &= f_y(\hat{x}'(s_i), \hat{x}''(s_i), \hat{y}'(s_i), \hat{y}''(s_i)) \text{ for } i = 0, 1, \dots, N \end{aligned} \quad (5.8)$$

The boundary conditions (5.6) and the collocation equations (5.8) give a nonlinear system of $2N + 10$ equations in $\{\alpha_i, \beta_i\}$. One can use Newton's method to solve the system for $\{\alpha_i, \beta_i\}$, and then the solution is approximated by equations (5.7).

Since each basis function $\phi_i(s)$ lives only on $[s_{i-3}, s_{i+3}]$ and is zero at the end points s_{i-3} and s_{i+3} for continuity, the j th collocation equation in (5.8) involves only $\{\alpha_k, \beta_k\}_{k=j-2, j-1, j, j+1, j+2}$. In the context of Newton's method, this results in a sparse matrix for fast linear algebra solvers. Precisely, for each element, there is a row in the Jacobian matrix with only ten nonzero entries.

To illustrate, we define a vector \mathbf{z} that encapsulates the unknowns $\{\alpha_i, \beta_i\}$:

$$\begin{aligned} \mathbf{z}(i) &:= \alpha_{i-3} \text{ for } i = 1, \dots, N+5 \\ \mathbf{z}(N+5+i) &:= \beta_{i-3} \text{ for } i = 1, \dots, N+5. \end{aligned} \quad (5.9)$$

To set up the Jacobian of the linear system using Newton's method, we loop over all elements for $i = 0, 1, \dots, N$ and put the entries of the Jacobian into the appropriate column of the matrix with a fixed row i . Since each element involves only 5 nonzero basis functions, the sparsity of the Jacobian will appear as in Figure (8). Each row (an equation for each element) has five adjacent entries. The bottom 8 rows in the matrix correspond to the boundary conditions where we fix the end points and the tangent vector at $s = 0, 1$. Again, each boundary condition equation involves only 5 entries due to the small intervals on which the basis functions are nonzero. The amenable sparsity structure in Figure (8) illustrates the efficiency in solving the system (5.5) using collocation with quintic B-splines that live on small intervals. Once we solve for \mathbf{z} , the system is approximated via equations (5.7). We perform Newton iterations on the nonlinear system of equations represented in equations (5.6) and (5.8) until the norm of the vector containing the computed value of the equations (which should be zero) is below a certain tolerance. That is, we can ensure that the approximate solutions in equations (5.7) satisfy the ODE and boundary conditions in equations (5.5) and (5.6) up to a desired tolerance. Damped Newton iterations might be required. In less than 30 seconds, a three dimensional system with 100 grid points can be solved. But, this of course depends on the quality of the initial guess for the solution.

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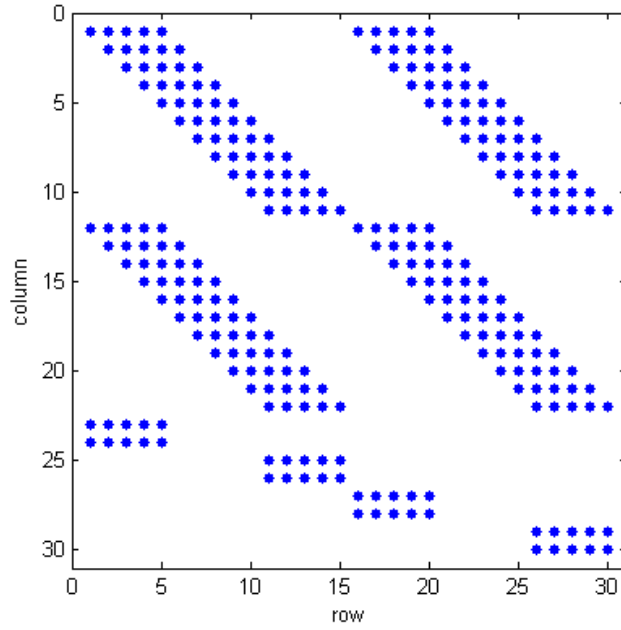


Figure 8: The sparsity structure of the Jacobian of the linear system for $\{\alpha_k, \beta_k\}$. Each nonzero entry is marked. Here, $h = 0.1$. The first $N + 1$ rows correspond to the $x(s)$ differential equation. The next $N + 1$ rows correspond to the $y(s)$ ODE. The last eight rows correspond to the boundary conditions. The matrix columns correspond to the vector z in equation (5.9). Note that with a rearrangement of rows, the matrix is banded.