The geometry of Euler's equations

lecture 2

A Lie group G as a configuration space

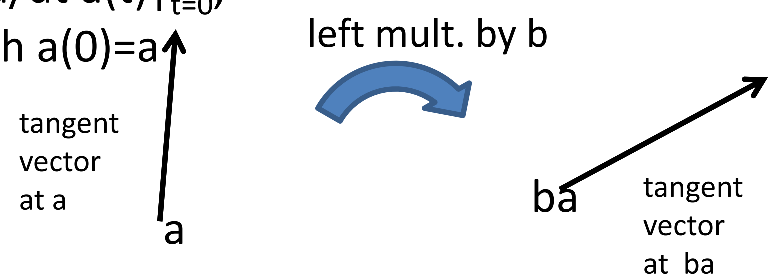
Subgroup of $N \times N$ matrices for some N

G … finite dimensional Lie group - we can think of it as a matrix group

Notation: a,b,… elements of the group

Basic point: we can move objects around (vector fields, forms, etc.) by left (or right) multiplication.

$\xi = d/dt\, a(t)|_{t=0}$, with $a(0)=a$



$b \cdot \xi = d/dt\,(b\, a(t))\,|_{t=0}$

for the right multiplication we can define $\xi \to \xi \cdot b$ in a similar way

co-vectors ($\sim$ forms): the dual of ($\xi \to b \cdot \xi$) moves forms from $T^*_{ba}\,G$ to $T^*_a\,G$

$\alpha \in T^*_{ba}\,G \to b^* \cdot \alpha \in T^*_a\,G$ (abusing the notation: b* is not the adjoint matrix)

The tangent space at the unit element: $\mathfrak{g}$ (the Lie algebra of G)

The adjoint action of G on $\mathfrak{g}$:

$$\xi \to a \cdot \xi \cdot a^{-1} = \mathrm{Ad}(a)\, \xi$$

Lie bracket on $\mathfrak{g}$:
Let $\eta \in \mathfrak{g}$ and $\dot{b}(t)|_{t=0} = \eta$. Then

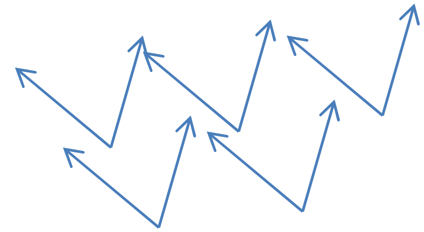
$$[\eta, \xi] = d/dt|_{t=0} \mathrm{Ad}\,(b(t))\, \xi$$

the group operation induces the Lie algebra structure

Structural constants of $\mathfrak{g}$:
$e_1, \ldots, e_n$ basis of $\mathfrak{g}$, $[e_i, e_j] = c_{ij}^k e_k$

Jacobi identity $[\xi, [\eta, \zeta]] + [\eta, [\zeta, \xi]] + [\zeta, [\xi, \eta]] = 0$

Coordinates on TG and T*G:



move the basis e_1, …, e_n by the left multiplication to each point of the group

get a frame of left invariant vector fields, still denoted e_1,…, e_n

Identify TG with $G \times \mathfrak{g}$ or with $G \times R^n$ by using the frame.

$$(a, \xi) \in G \times \mathfrak{g} \to a \cdot \xi \in T_a G$$

or

$$(\, a\, ;\ \xi^1, \ldots, \xi^n) \to a \cdot (\xi^i e_i) \in T_a G$$

Similarly, T^*G is identified with $G \times \mathfrak{g}^*$ or with $G \times R^n$

$$(a, \alpha) \in G \times \mathfrak{g}^* \to (a^*)^{-1} \cdot \alpha \in T_a^* G$$

or

$$(\, a \,;\, y_1, \ldots, y_n) \in G \times R^n \to (a^*)^{-1}(y_j e_j^*) \in T_a^* G$$

dual basis to $e_1, \ldots, e_n$

The coordinates $y_1, \ldots, y_n$ can also be thought of as functions on T^*G.

Prolongation of the action of G on itself by the left multiplication to T^*G:

$$b \cdot (\, a \,;\, y_1, \ldots, y_n) \quad \to \quad (\, ba \,,\, y_1, \ldots, y_n)$$

Note that $y_1, \ldots, y_n$ considered as functions on T*M are invariant under (the prolongation of) the left multiplication. This is more or less by definition.

Interpretation:
T^*G … we follow
$a(t)$ and $d/dt\, y(t)$ (momentum)

$C^\infty(G \setminus T^*G)$ are exactly the functions of $y_1, \ldots, y_n$.

$Y = G \setminus T^*G$ reduced phase space:
we "forget" $a(t)$ and follow only $d/dt\, y(t)$

In fact, in this case $Y = G \setminus T^*G$ can be identified with R^n or $\mathfrak{g}^*$

The functions $y_1, \ldots, y_n$ provide coordinates on Y (which are global)

$C^\infty(Y)$ inherits a Poisson bracket from T^*M. What is the bracket?

equations for y:
$d/dt\, y(t) = \{H,y\}$

hamiltonian
$H=H(y)$

enough to calculate $\{y_i, y_j\}$

This needs a little work, the main point is to use the formula
$de^*_j(e_k, e_l) = -e^*_j([e_k, e_l])$
for the invariant forms/fields
-see next

Calculation: $\{y_l, y_j\} = c^k_{ij}\, y_k,$

where c^k_{ij} are the structure constants of the Lie algebra $\mathfrak{g}$

Canonical form of T*G

e_1*, …, e_n* frame of the left-invariant forms, dual to the frame e_1,…,e_n

Recall the natural coordinates on T*G:

$(\,a\,;\,y_1, \ldots , y_n) \in G \times R^n \qquad \leftrightarrow \qquad y_j\, e^*_j$

The canonical 1-form on T*G: $\alpha = \mathbf{y_j\, e^*_j}$

The symplectic form: $\omega = \mathbf{d}\alpha = dy_j \wedge e^*_j + y_j\, de^*_j =$

$$= \mathbf{dy_j \wedge e^*_j - y_j c^j_{kl} e^*_k \wedge e^*_l}$$

where we used the Cartan formula

$d\alpha(\xi,\eta) = \xi\cdot \alpha(\eta) - \eta\cdot \alpha(\xi) - \alpha([\xi,\eta])$ with $\alpha = e^*_p$, $\xi = e_q$, $\eta = e_r$

= 0 in our case

$[e_p, e_q] = c^r_{pq}\, e_r$

In the local frame in T*G given by $y_1,\ldots,y_n$, $e_1,\ldots,e_n$ the form omega is given by the matrix

$$\begin{pmatrix} 0 & I \\ -I & -C(y) \end{pmatrix}$$

with notation $y_k\, c^k_{\ ij} \sim C(y)$

The inverse ω^{rs} of this (anti-symmetric) matrix is given by

$$\begin{pmatrix} -C(y) & -I \\ I & 0 \end{pmatrix}$$

The Poisson bracket is (in our conventions) $\{f,g\} = \omega^{rs} f_s\, g_r$

In particular, $\{y_i , y_j\} = c^k_{\ ij}\, y_k$, Remark:

Note that the group G was any Lie group. We have shown:

If G is a Lie group and $\mathfrak{g}$ is its Lie algebra with structural constants c_{ij}^k, then on the dual $\mathfrak{g}^*$ of $\mathfrak{g}$ the formula

$$\{y_i, y_j\} = c_{ij}^k y_k$$

defines a Poisson bracket on $\mathfrak{g}^*$.

This was already known to S. Lie, but he did explore the implications.

In the 1960s this structure was used by A. A. Kirillov to obtain important results in representations of nilpotent groups, and to develop his "method of orbits".

$b \in G \quad \Rightarrow \quad$ the left shifts $\quad a \to ba \quad$ extend to a symplectic diffeom. of T^*G:

represent the form $y_i\, e^*_i$ at a

in the coordinates $(a, y_1, \ldots, y_n)$: **$b \cdot (a, y_1, \ldots, y_n) = (ba, y_1, \ldots, y_n)$**

the infinitesimal version of these deformations:

$$\xi \in \mathfrak{g} \text{ generates an infinitesimal symplectic deformation of } T^*G$$

$ba = (1+\epsilon\, \xi)\, a$

y does not change, since the coordinate functions y_i are invariant under the left shifts

$(a, y) \to (a + \epsilon\, \xi \cdot a, y)$

Recall: Infinitesimal sympl. defs. $\longleftrightarrow$ **functions** (perhaps modulo corrections)

What is the generating function f for **$(a, y) \rightarrow (a+\epsilon\,\xi\cdot a, y)$?**

In the coordinates of lecture 1 it would be $f = p_i\,\xi^i$

We need to express this in the presently used coordinate frame $y_1,\ldots, y_n, e_1, \ldots, e_n$

coordinates of the infinitesimal deformation in this frame:

first coordinate

$(0, \ldots, 0, [Ad(a^{-1})\,\xi]^1, \ldots, [Ad(a^{-1})\,\xi]^n)$

no shift in y

need to use this form because our coordinate frame is left invariant (not right-invariant).

arises from $\xi\cdot a = a\; a^{-1}\cdot\xi\cdot a = a\cdot Ad(a^{-1})\xi$

recall f = $p_i \xi^i$ in coordinates of lecture 1

Recall **Noether's Theorem:**

The generating function of the infinitesimal symplectic transformation above will be

$$f(a,y) = \langle y, Ad(a^{-1})\xi \rangle = \langle Ad(a^{-1})^* y, \xi \rangle$$

Moreover, **f(a,y) will be conserved for any Hamiltonian depending only on y**

Exercise: check "by hand" that $\{ f , y_i \} = 0$ for each i

Example: rigid body rotation

y momentum in the coordinate frame moving with the body

$Ad(a^{-1})y$ momentum in the coordinate frame fixed in space

Definition:

$$M(a,y) = Ad(a^{-1})^* y \in \mathfrak{g}^*$$

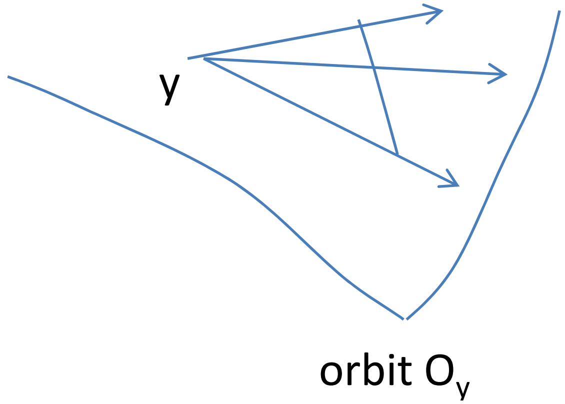
is called the **moment map.**

For any Hamiltonian depending only on y the evolution preserves M, or **dM/dt = 0**.

This is because the ξ above can be taken as any element of $\mathfrak{g}$

determined by $C_i(y)=c_i$, C_i the Casimir functions, $\{C_i, y_j\}=0$ for each $j=1,\ldots,n$

"Symplectic leaves" of the Poisson manifold $\mathfrak{g}^*$ (a connected G)



A subspace through y generated by all possible vectors $dy/dt=\{H,y\}$, as H runs through all $H=H(y)$ is contained in the tangent space to the orbit $O_y = \{Ad^*(a)y,\ a\in G\}$ (because of the conservation of M for any such H, for example).

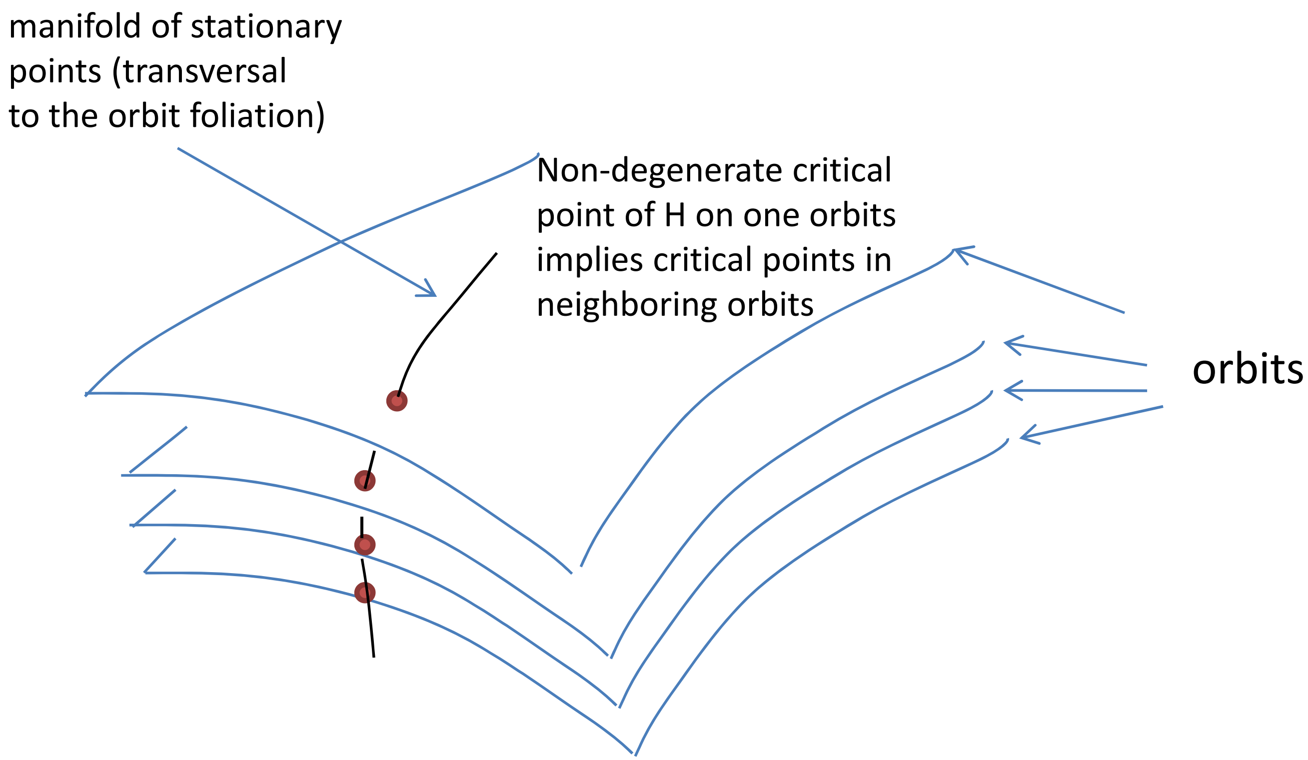
Vice versa, any vector in $T_y\, O_y$ can be obtained in this way.

Hence: **Symplectic leaves** $\longleftrightarrow$ **orbits**

(See A.A.Kirillov's book for more and implications to representations.)

Hence the evolution on $\mathfrak{g}^*$ given by H(y) and $\{\ ,\ \}$ really describes a family of hamiltonian systems (parametrized by the orbits)

Example: non-degenerate stationary points are (locally) parametrized by the orbits:



Example: the simplest non-commutative group G

(Commutative G leads to $\{y_i, y_j\}=0$)

$$G = \{\ A = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}\ ;\ a > 0, b \in R\}$$

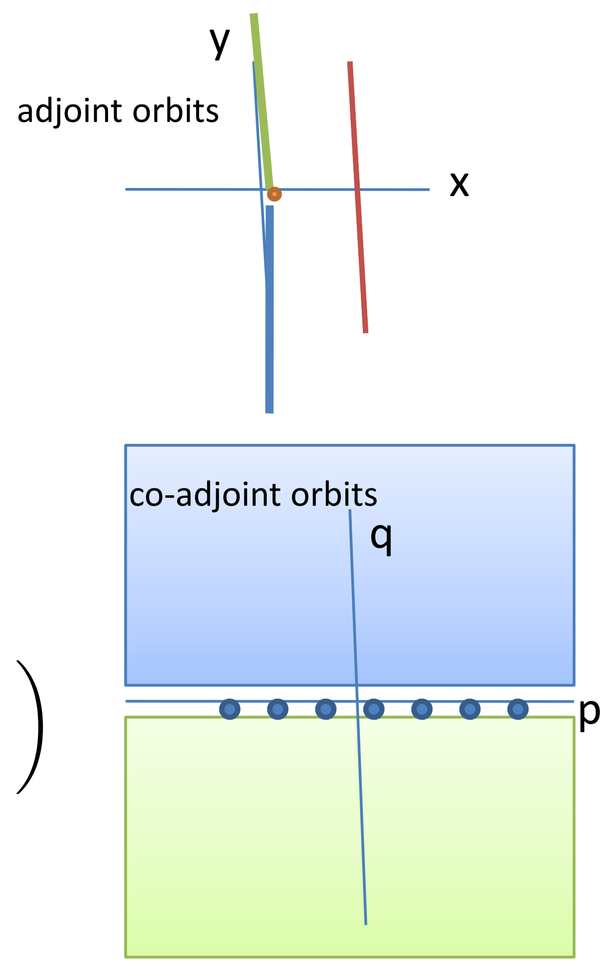
the group of orientation-preserving affine transformation of R (maps of the form x $\rightarrow$ ax+b) with a>0

$$\mathfrak{g} = \{\, X = \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix}\ ;\ x, y \in R\}$$

$$\mathfrak{g}^* = \{\, P = \begin{pmatrix} p & q \\ 0 & 0 \end{pmatrix}\ ;\ p, q \in R\}$$

$$< P, X >= px + qy$$

$$\mathrm{Ad}(A)\ \sim\ \begin{pmatrix} 1 & 0 \\ -b & a \end{pmatrix} \qquad \mathrm{Ad}(A)^*\ \sim\ \begin{pmatrix} 1 & -b \\ 0 & a \end{pmatrix}$$



Left invariant geodesics

given by Hamiltonian

$H=H(p,q)=(p^2+q^2)/2$

note the change of notation – both p and q are components of the momentum

$\{p,q\} = q$

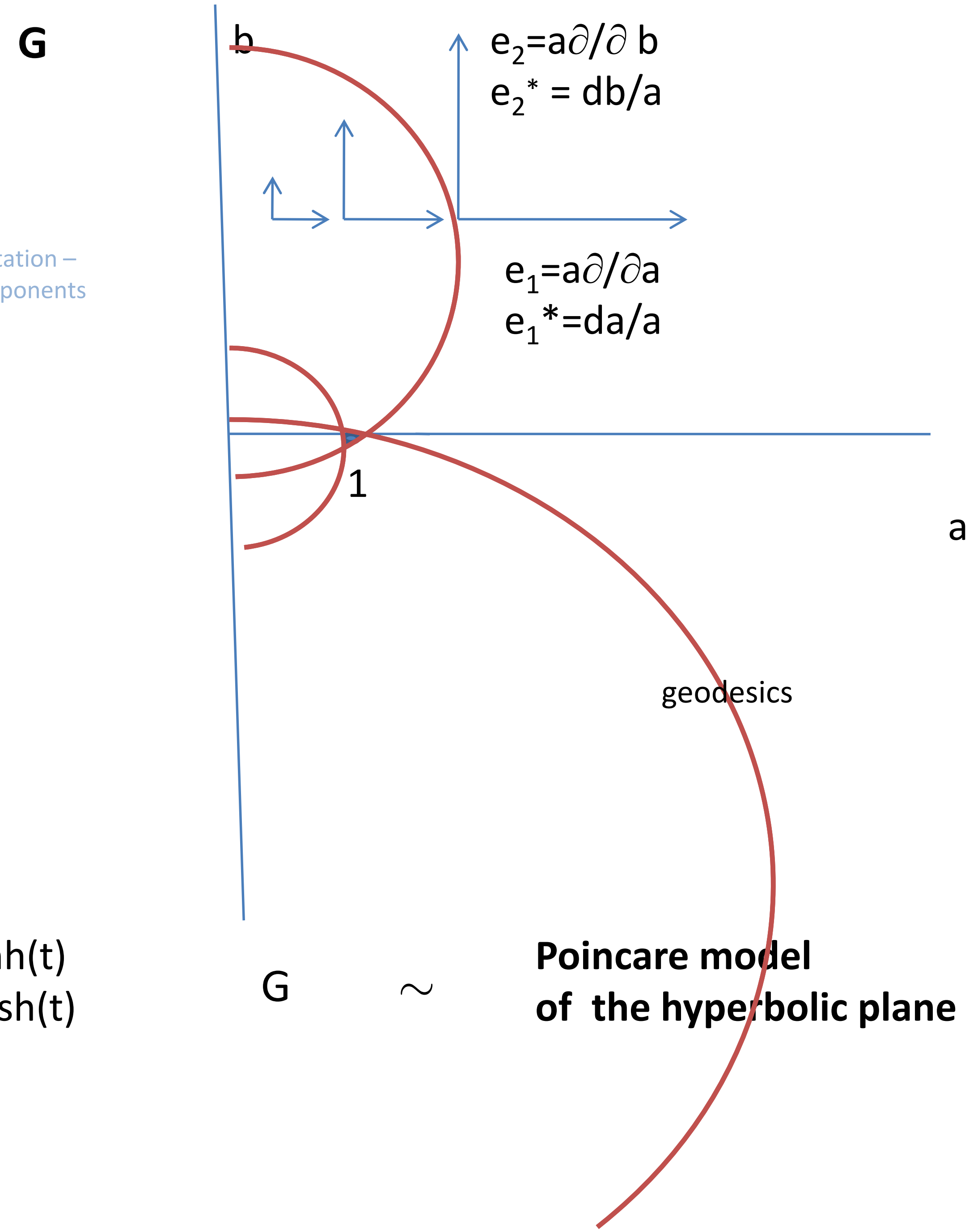
$dp/dt = \{H,p\} = q^2$

$dq/dt = \{H,q\} = - pq$

Momentum conservation:

$da/dt = pa, \quad db/dt = qa$

Example of solutions: $a(t) = a_0 \tanh(t)$
$b(t) = a_0/\cosh(t)$



G $\sim$ **Poincare model of the hyperbolic plane**

Remark: Direct Lagrangian approach (see also V.I.Arnold's book)

L left-invariant Lagrangian on TG

a(t) curve in G

$v(t) = a^{-1}\, da/dt$ velocity in the coordinates of the left-invariant frame

$\int_{t_1}^{t_2} L(v(t))\, dt$ action

$a_\epsilon(t) = a(t) b_\epsilon(t)$ with $b_\epsilon \sim 1+\epsilon\xi$ "nearby trajectory"

$v_\epsilon = a_\epsilon^{-1}\, da_\epsilon/dt = \mathrm{Ad}\,(b_\epsilon^{-1})\, v + b_\epsilon^{-1} db_\epsilon/dt$ velocity of the nearby trajectory

$v_\epsilon = v + \epsilon\,(-[\xi,v] + d\xi/dt) + O(\epsilon^2)$

$L(v_\epsilon) = L(v) + \epsilon L_v(-[\xi,v] + d\xi/dt) + O(\epsilon^2)$

$\int_{t_1}^{t_2} (L(v_\epsilon) - L(v))\, dt = -\epsilon \int_{t_1}^{t_2} (<dL_v/dt, \xi> + <L_v, [\xi,v]>)\, dt + O(\epsilon^2)$

$F(v)$ defined by $<F(v),\xi> = -<L_v, [\xi,v]>$, or (abusing notation), $F(v) = \mathrm{Ad}^*(v)\, L_v$

Equation for v: $dL_v/dt = F(v)$ (does not involve a(t); recovering a: $da/dt = a(t)\, v(t)$)

Other calculable geodesics for left-invariant metrics:

Sl(2,R) $\sim$ motion of a rigid body in the hyperbolic plane

$\mathbf{H_3}$ tree-dimensional Heisenberg group, Caratheodory metrics

$\mathbf{S^3}$ or **SO(3)** $\sim$ three-dimensional rigid body (Euler's equations for rotating bodies)

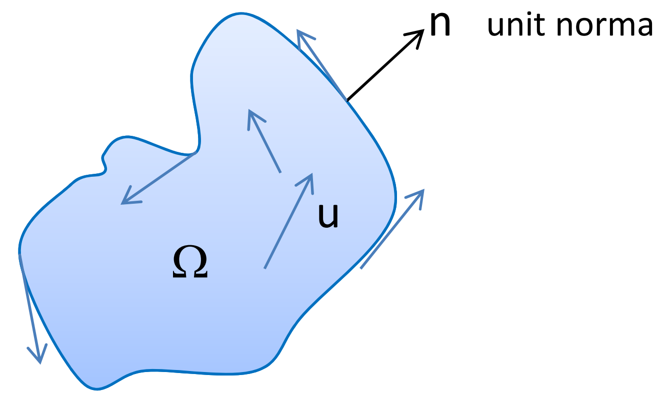
and more ….

We expect: 3d group with a left-invariant hamiltonian
$\Rightarrow$ co-adjoint orbits have dimension at most 2
$\Rightarrow$ equations for geodesics solvable by quadratures

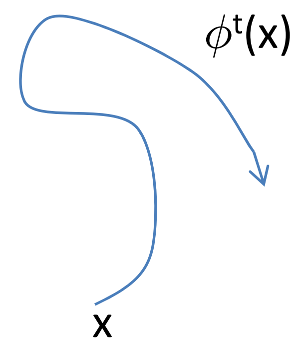
On the other hand: 6d groups (such as **Sl(2,C)** or **SO(4)**) – will often have 4d orbits – potential for "chaos" in the reduced equations

Special Hamiltonians can still give integrability:
Kovalevskaya's top, n-dim rigid body in Euclidean space (Manakov)

Ideal Incompressible Fluids



Configuration space: $G=\mathrm{Diff}_0(\Omega)$ volume-preserving diffeomorphisms of Ω (group under map composition)



Motion the fluid: curves in G, $t \rightarrow \phi^t$

$t \rightarrow \phi^t(x)$ trajectory of a "particle"

Symmetries: ϕ^t is a solution, $\psi \in G$ $\Rightarrow$ $\phi^t \circ \psi$ is a solution

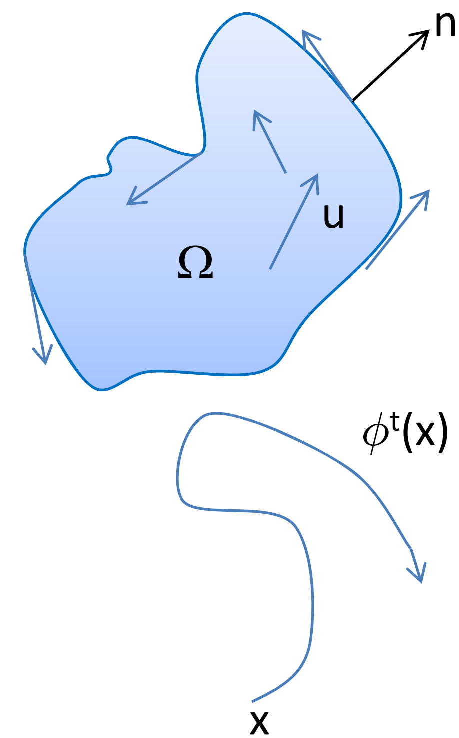
"particle re-labeling"

Hamilton's principle: the actual fluid motions "solutions"

are extremals of

$$\int_{t_1}^{t_2} \int_{\Omega} | \, \dot{\phi}^t(x)|^2 \; dx \; dt$$

among curves ϕ^t in G with ϕ^{t_1} and ϕ^{t_2} fixed



This leads to Euler's equations for

$$u(x,t) = \dot{\phi}(\; \phi^{-1}(x,t) \; , \; t \;)$$

we are now using right shifts,
rather then the left shifts….

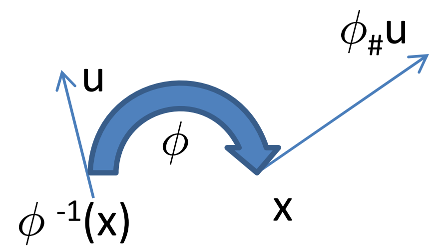
It's not hard to get the equation by direct calculation – but we will follow the Hamiltonian approach, which also immediatelly gives the conservation laws coming from the invariance by G ("particle re-labelings")

The Lie algebra of $G=\mathrm{Diff}_0(\Omega)$ and its dual

$U = T_{identity}\,G$ ……… the Lie algebra of G $\sim$ div-free vector fields in Ω tangent to the boundary

$u,v \in U$ …………$[u,v]$ the usual Lie bracket of u,v (is exactly the bracket induced from the group)

Adjoint action: $Ad(\phi)\, u = \phi_\# u$ $\quad (\phi_\# u)(x) = D\phi\,(\phi^{-1}(x))\; u(\phi^{-1}(x))$



The dual U* linear functionals on U can be obtained most naturally by

$$u \quad \to \quad \int_\Omega a_i(x)\, u^i(x)\, dx = \langle a,u \rangle$$

where a is viewed as a 1-form.

The co-adjoint action (of $G=\mathrm{Diff}_0(\Omega)$ on U^*)

$\langle a, \phi_{\#}u\rangle = \langle \phi^* a , u\rangle$

$\phi^* a$ ...... the usual pull-back

$[\, \phi^* a\, (\phi(x))\,]_i \;=\; a_j(x)\, \phi^j_{,i}(x)$

$\mathrm{Ad}^*(\phi)\, a = \phi^* a$

So it seems natural to identify U* with one-forms, but the problem is that the correspondence is not one-to-one:

$a = df \quad \Rightarrow \quad \langle a,u\rangle = 0$ for each $u \in U$: $\quad \int_\Omega \nabla f\; u = \int_\Omega - f\,(\mathrm{div}\, u) = 0$

To get a one-to-one correspondence

replace 1-forms by (1-forms)/(exact differentials df)

1-forms / differentials df $\xrightarrow{d}$ 2-forms $\xrightarrow{\text{via the volume element}}$ vector fields

$a_i\, dx^i$ $\qquad$ $a_{i,j}\, dx^j \wedge dx^i$ $\qquad$ $\omega^i\, \partial/\partial x^i$, with $\omega^i = \epsilon^{ijk} a_{k,j}$

Or, in the "vector calculus" notation:

$a \sim (a_1, a_2, a_3) \longrightarrow$ $\boldsymbol{\omega}$ **= curl a**

Action of $\mathrm{Diff}_0\,(\Omega)$

in the "a-coordinates"

$Ad^*(\phi)\, a = \phi^* a$

In the "ω-coordinates"

$\mathbf{Ad^*(\phi)\omega = \phi_{\#}\omega}$

Duality in between U and U* in the ω-coordinates on U*

$a \sim a_i\, dx^i \qquad da \sim \mathrm{curl}\, a \sim \omega \sim \omega^i\, \partial/\partial x^i$

vector potential of u, analogue of the 2d stream function

$u \in U$, $u = \mathrm{curl}\, \Psi$, $\mathrm{div}\, \Psi = 0$ in Ω, $\Psi \wedge n = 0$ on $\partial\Omega$ (when Ω is topologically trivial)

for simplicity you can also think of $\Omega = R^3$

$\int_\Omega a\, u\, dx = \int_\Omega \omega\, \Psi\, dx$ (Check that the boundary term vanishes due to the boundary conditions)

In these "coordinates" the basic variable is $\omega = \omega(x,t)$ ("vorticity")

The vector fields $u \in U$ are identified with their vector potentials Ψ

The Hamiltonian = kinetic energy : $\omega \in U^*$, $\mathrm{curl}\, u = \omega$, $\mathrm{div}\, u = 0$, $un|_{\partial\Omega} = 0$,
$-\Delta \Psi = \omega$, $\mathrm{div}\, \Psi = 0$, $\Psi \wedge n|_{\partial\Omega} = 0$

$$H = H(\omega) = \int_\Omega \tfrac{1}{2} |u|^2\, dx = \int_\Omega \tfrac{1}{2} \omega \Psi\, dx$$

Poisson bracket

Lie bracket in U in terms of the vector potentials Ψ

$u = \text{curl}\, \Psi$, $v = \text{curl}\, \Phi \in U$, $w = [u,v]$, $w = \text{curl}\, \Theta$

$w = [u,v] = u\nabla v - v\nabla u = -\ \text{curl}\, (u \wedge v)$ (we used div u = div v = 0)

$\Theta = -\ \text{curl}\, \Psi \wedge \text{curl}\, \Phi$

Ψ and Φ can be considered as linear function on U*

$l_\Psi : \omega \to \int_\Omega \omega\Psi$ and $l_\Phi : \omega \to \int_\Omega \omega\Phi$

Poisson bracket: $\{l_\Psi , l_\Phi\} = l_\Theta = l_{-\,\text{curl}\Psi \wedge \text{curl}\,\Phi}$

general functionals: $\{ F , G \} (\omega) = \int_\Omega \omega \left(-\ \text{curl}\, (\delta F/\delta\omega) \wedge \text{curl}(\delta G/\delta\omega) \right) dx$

"variational derivative", expressing F' and G' using the duality with U (need some smothness)

Example: $H(\omega) = \int_\Omega \frac{1}{2}\, \omega\, \Psi$ (the Euler Lagrangian introduced earlier)

$\delta H/\delta\omega = \Psi$ (with $-\Delta\Psi = \omega$ + boundary cond.)

ω = curl u, div u = 0 + boundary cond., u = curl Ψ

d/dt $l_\Phi(\omega) = \int_\Omega \omega_t \Phi = \{ H , l_\Phi\}(\omega) = \int_\Omega \omega(-$ curl $\Psi \wedge$ curl $\Phi) = \int_\Omega$ curl$(\omega\wedge u)\, \Phi$

evolution of the "coordinate" given by Φ

Analogous to the finite-dimensional equation $dy_i/dt=\{H,y_i\}$

Φ is arbitrary div-free with $\Phi\wedge n = 0$ at the boundary

ω_t + **curl (u$\wedge\omega$) = 0** or

vorticity formulation of Euler's equations

ω_t **+ [u , ω] = 0**

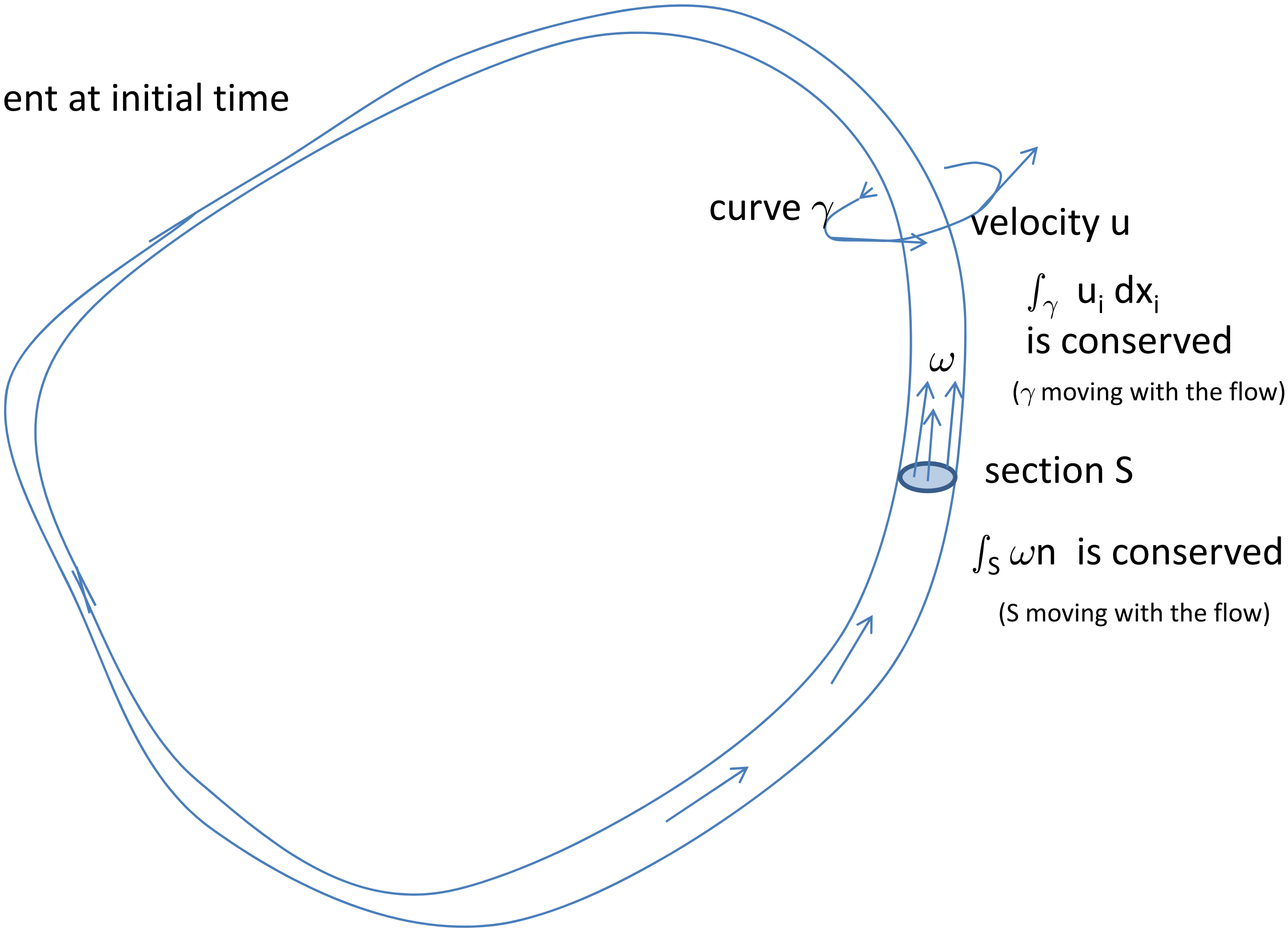
Noether's theorem (conservation of the moment function)

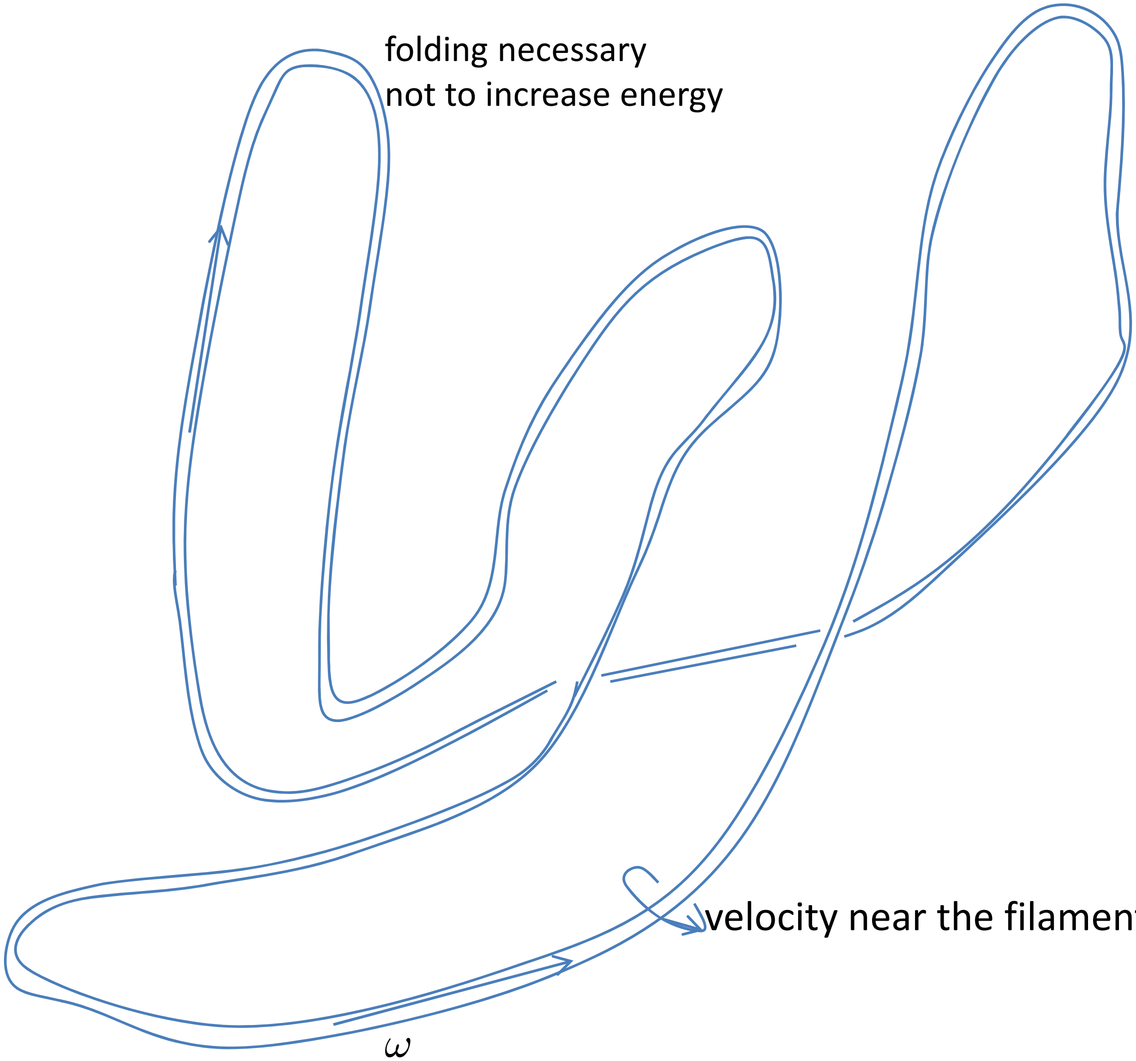
$Ad^*(\phi^t)^{-1}\, \omega(t) = \omega(0)$ or $\boldsymbol{\omega(t) = \phi_\# \omega(0)}$

Helmholtz's law
"vorticity moves with the flow"

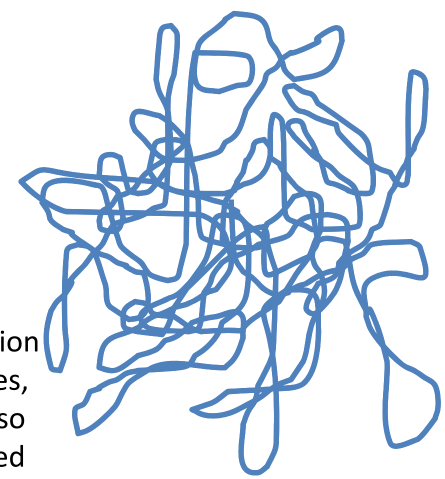
Consequences of Helmholtz's law

Vortex filament at initial time





eventually it will presumably become very complicated



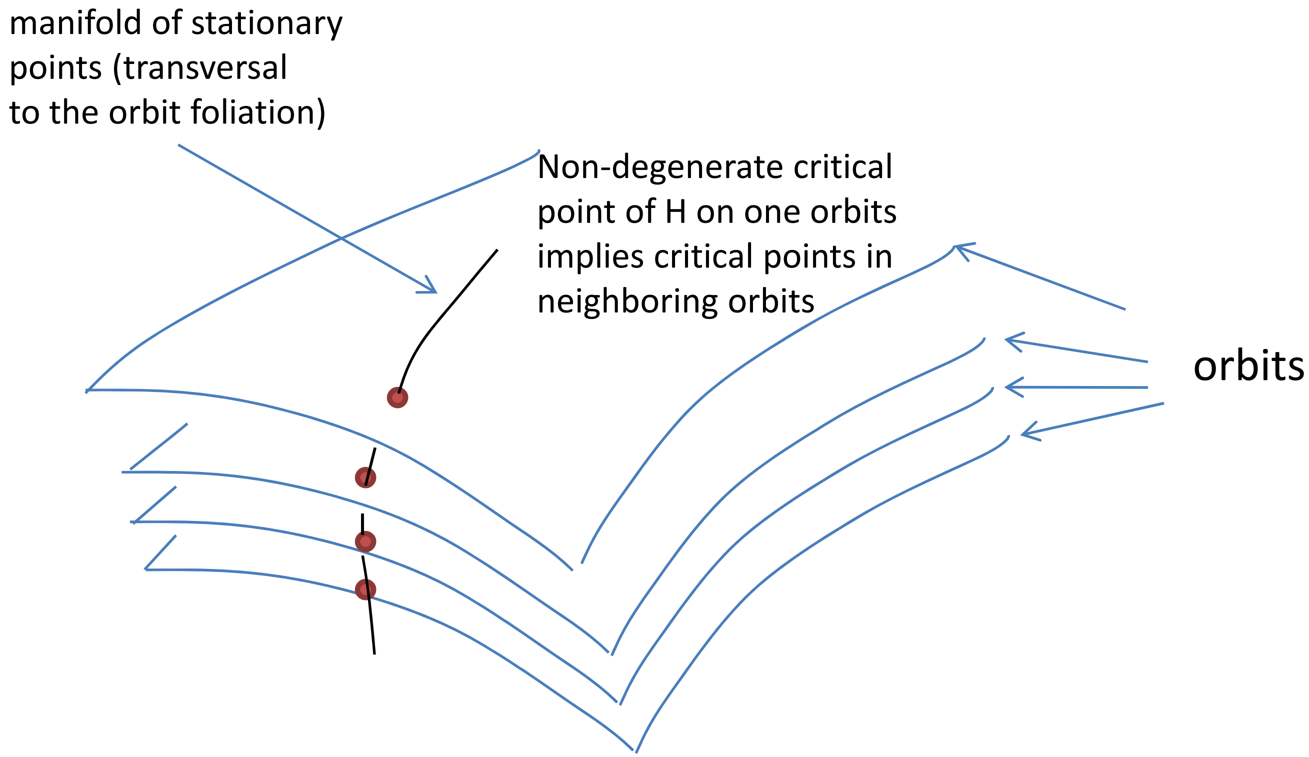
A benefit of the co-adjoint orbit approach:

some “reduced” systems (such as point vortices, vortex fillaments,…) come with natural hamiltonian structure.

1) 2d ….. sets of a given number of points can be moved around by $\mathrm{Diff}_0(\Omega)$ - finite dimensional orbits – finite dimensional Hamiltonian system - “point vortices” (Hamiltonian can be taken from the original Euler’s eq. if we remove the infinite “self-energy” of each vortex.

2) 3d ….. curves can be moved around by $\mathrm{Diff}_0(\Omega)$
get a natural symplectic structure on the “manifold of curves”
for certain “weak filaments” energy $\sim$ length
curves with the sympl. structure, Hamiltonian = length,
- flow of curves by binormal curvature

Geometric picture of the steady-states for 2d Euler:



Can be established rigorously in 2d under some (reasonable) assumptions
(A. Choffrut, V.S.)

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