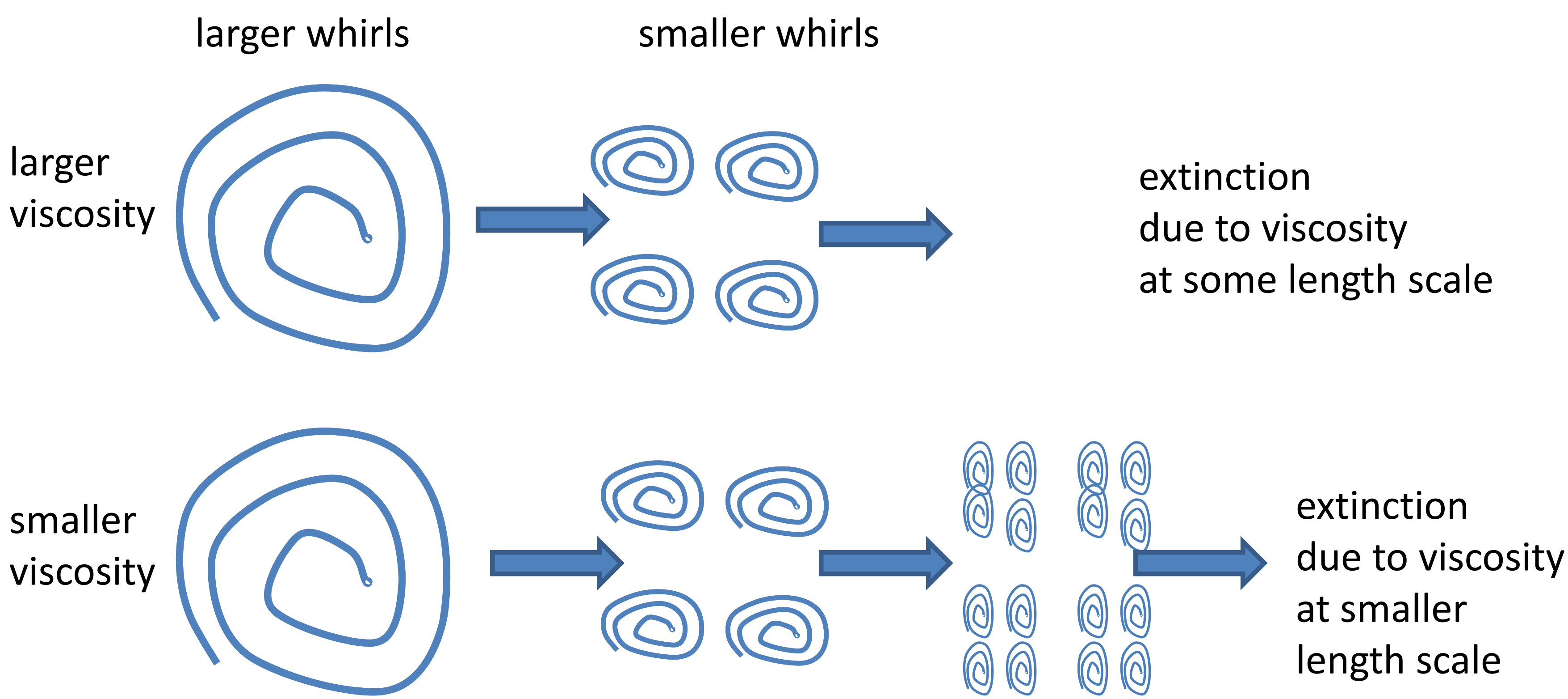
Kolmogorov’s 1941 Theory

Introduction

Richardson’s idea of the independence of the macroscopic flow features on viscosity

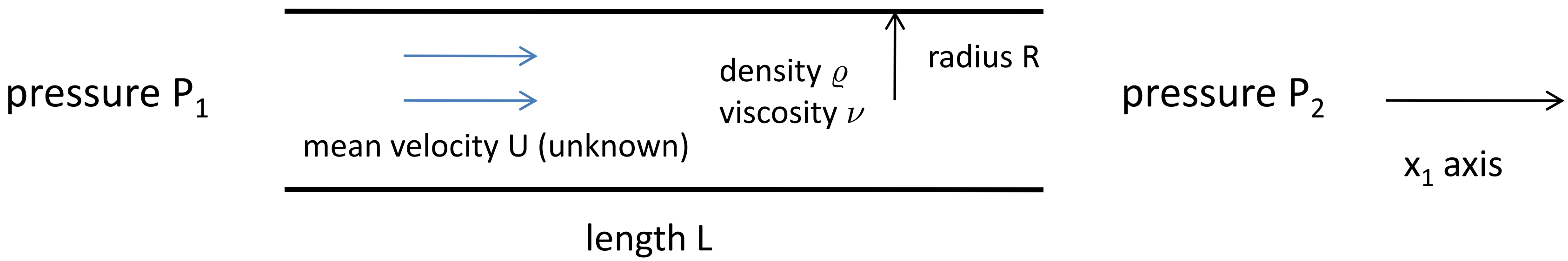
cartoon picture (in reality it is important that the flow is 3D)



The net macroscopic effect remains (approximately) the same

$$-\nu \Delta u + u\nabla u + \nabla p/\varrho = 0$$
$$\text{div } u = 0$$
boundary condition $u = 0$

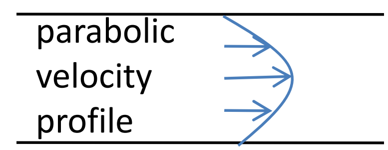
Let's apply it to pipe flows:



Problem: How much fluid flows through the pipe per unit of time?

Or: what is the mean velocity of fluid U in the pipe (defined by $\pi R^2 U$ = amount of fluid per unit time)?

Attempt #1: <u>Explicit solution of Navier-Stokes!</u>

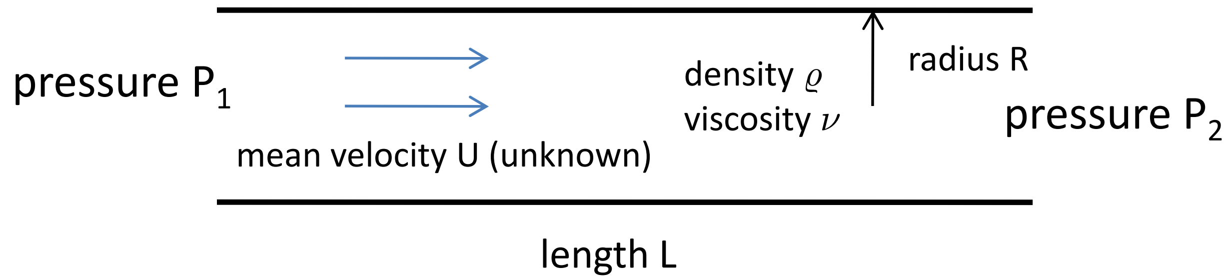


$$\begin{aligned} u_1 &= 2U(1-(x_2^2+x_3^2)/R)\,,\ u_2 = u_3 = 0 \\ p &= 8\varrho\nu U x_1/R^2 \ + \ \text{const.} \\ U &= (P_1-P_2)R^2/8\rho\nu L \end{aligned}$$

"Poiseuille's flow"

Only works for real flows when **Re=UR/ν** is below a few thousand; recall $\nu = 10^{-6}$ for water

$Re = UR/\nu$



Richardson's philosophy: U should be independent of ν once Re is large

(and the flow becomes turbulent)

Available quantities, assuming translational symmetry in x_1:

$P' = (P_1 - P_2) / L$ (pressure drop per unit length), R, ρ, (but not ν !)

Task: express U from these quantities in a way which is independent of the choice of units of measurement.

dimensions: $[P']=[\text{force}][\text{length}]^{-3} = [\text{mass}][\text{length}]^{-2}[\text{time}]^{-2}$

$[R]=[\text{length}]$

$[\varrho]=[\text{mass}][\text{length}]^{-3}$

$[U]=[\text{length}][\text{time}]^{-1}$

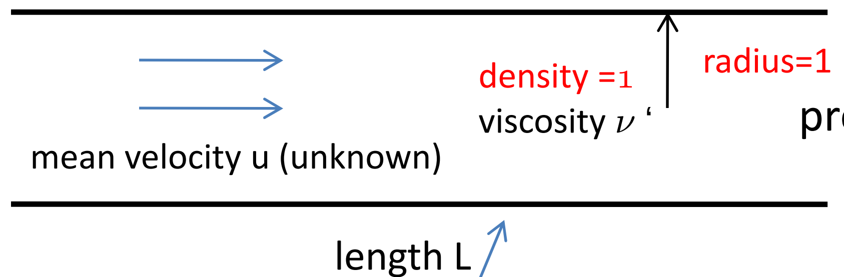
Compare with the previous formula $U = P'R^2/8\varrho\nu$

The only possibility: **$U = \text{const.}\,(P'/\varrho)^{1/2}\,R^{1/2}$**

In practice the formula works quite well – a remarkable fact given the rough derivation! Corrections to the formula: c=c(Re, roughness coefficient of the wall), but it does become almost constant for large Re and a given wall roughness. It seems to be absolutely hopeless to try to derive this from Navier-Stokes.

Another way
to look at
the derivation

pressure =L
so that pressure
per unit length = 1



pressure =0

Normalize by suitable choice of units of length, time, mass;
then u should be independent of ν ‘ (which plays the role of 1/Re’)

In our set-up Re’ would differ by a constant factor from the standard Re=RU/ν

Go back to the original units and recover $U = \text{const.}\ (P'/\varrho)^{1/2}\ R^{1/2}$
with const. = u

The same argument in a slightly different language:

1. Assume the formula $\mathbf{U = f(P',R,\varrho)}$ is independent of the choice of units.

2. Change units: [old length]= λ [new length]
   [old time] = τ [new time]
   [old mass] = μ [new mass]

3. Then [P'] = [force]/[length] ^{3} = [mass] [length] $^{-2}$ [time] $^{-2}$
   and hence [old P'] = $\mu\,\lambda^{-2}\,\tau^{-2}$ [new P']
   similarly [old R] = λ [new R]
   [old ϱ]= $\mu\,\lambda^{-3}$ [new ϱ]
   and [old U]= $\lambda\,\tau^{-1}$ [new U]

4. The formula U=f(P',R, ϱ) is the same in the old and the new quantities
   therefore $\mathbf{f(\mu\lambda^{-2}\tau^{-2}P', \lambda R, \mu\lambda^{3}\varrho) = \lambda\,\tau^{-1} f(P', R, \varrho)}$
   which is the same as
   $$f(\alpha P', \beta R, \gamma\varrho) = \alpha^{1/2}\beta^{1/2}\gamma^{-1/2} f(P', R, \varrho)$$
   or
   $$\mathbf{U = f(P',R,\varrho) = P'^{1/2}\varrho^{-1/2}R^{1/2}\ f(1,1,1)}$$

In terms of Navier-Stokes, the “normalization” is related to the **scaling symmetry:**

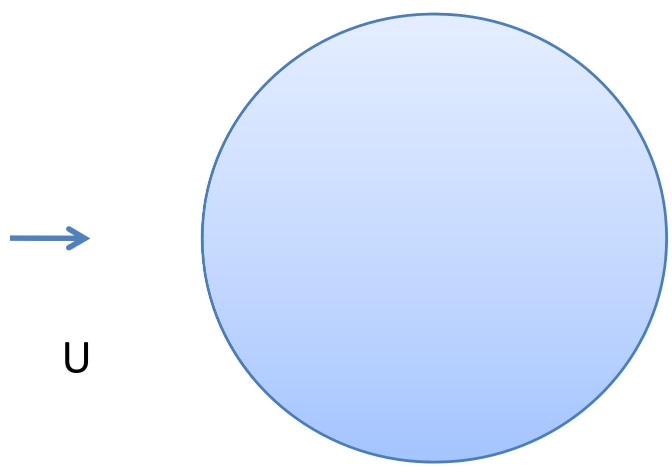
(which also plays a crucial role in the regularity theory)

If $u(x,t)$, $p(x,t)$ solve Navier Stokes with a given viscosity ν

$\longrightarrow$

for any $\lambda > 0$
$\lambda\, u(\lambda x, \lambda^2 t)$, $\lambda^2 p(\lambda x, \lambda^2 t)$
also solve Navier-Stokes
(with the same viscosity)

Not good for scale models (as can also be seen by using the Reynolds number):



1:10 scale model,
(corresponding to λ=10)

velocity should be 10 U
(can be hard to achieve in practice in some cases)
also, our **camera should run 100 times faster**

In reality, the situation is not so bad,
due to the Richardson effect.

Following Kolmogorov (1941), we introduce

ϵ = **rate of energy dissipation in the fluid per unit mass**

It is not the instantaneous dissipation (which would depend on time), but a suitable average (either over time, or a statistical ensemble). It can depend on x.

For example, in the pipe flows we can imagine the following:

$\epsilon = \epsilon$ (r) with r = distance to pipe axis

In fact, we can imagine that ϵ(r) be nearly constant in r, except possibly near the boundaries, when r$\sim$ R . In practice this is not a bad assumption. (We will look at the possible r-dependence later.)

$\epsilon = \epsilon$ (r) (with r = distance to pipe axis) R

U energy dissipation per unit mass

The variable ϵ should be considered as a "macroscopic quantity", similar to U, P', etc., in contrast to the "microscopic field" u(x,t). Typically we do not need all the microscopic details of u(x,t).

According to Kolmogorov, ϵ, together with the density ϱ are the basic "macroscopic" quantities characterizing the flow. The key point now is

The main assumption: **ϵ does not depend on ν**

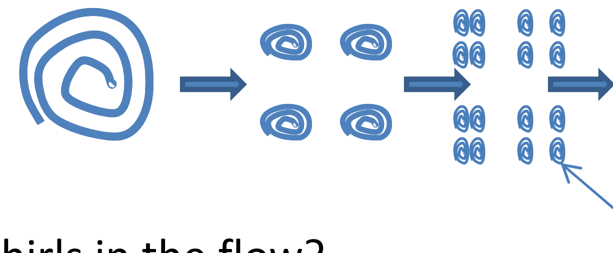
Typical considerations

1. Dependence of ϵ on U and R assuming ϵ is constant in r.

This could of course be also derived based on our previous formula **U = const. $(P'/\varrho)^{1/2} R^{1/2}$**

dimension of ϵ: [mass][velocity] ^{2} [time] $^{-1}$ [mass] $^{-1}$ = [length] ^{2} [time] $^{-3}$
dimension of R: [length]
dimension of U: [length][time] $^{-1}$
If $\epsilon = \epsilon$ (U,R), independently of the choice of units, we must have
ϵ = const. U^3/R

2. Kolmogorov Length:



What are the dimension λ of the smallest whirls in the flow?
Or: what is the space resolution λ we need for a full numerical simulation?
Or: what is the smallest length-scale λ in the flow?

Kolmogorov: The only relevant quantities are ϵ, ϱ, and ν

dimensions: $[\epsilon]$ = [length]^2 [time]$^{-3}$, $[\nu]$=[length]^2 [time]$^{-1}$, $[\varrho]$ = [mass][length]$^{-3}$

$[\lambda]$= [length]

The only possible formula for λ (assuming it exists):

$$\boldsymbol{\lambda = \text{const.}\ \epsilon^{-1/4}\, \nu^{3/4}} \quad - \quad \text{“Kolmogorov length”}$$

Recalling ϵ = const. U^3/R and Re = UR/ν , we obtain

$$\boldsymbol{\lambda = \text{const.}\ R\ /\ Re^{3/4}}$$ (for typical garden hose flows Re $\sim 10^5 - 10^6$)

3. Kolmogorov Time:

What are the smallest time-scales τ (or the highest time frequencies $f = 1/\tau$)?

Relevant variables: $\epsilon, \ \nu, \ \varrho$

Note that we do not list the macroscopic velocity U. Therefore we are talking about local time-scales, as observed in the system of a moving particle

Exercise: based on dimensional analysis, show that the only possible formula is

$$\boldsymbol{\tau = \text{const. } \epsilon^{-1/2} \nu^{1/2}}$$

In terms of Reynolds number

$$\boldsymbol{\tau = \text{const. } (R/U) \ \ Re^{-1/2}}$$

of order 10^{-5} sec in a typical garden hose flow

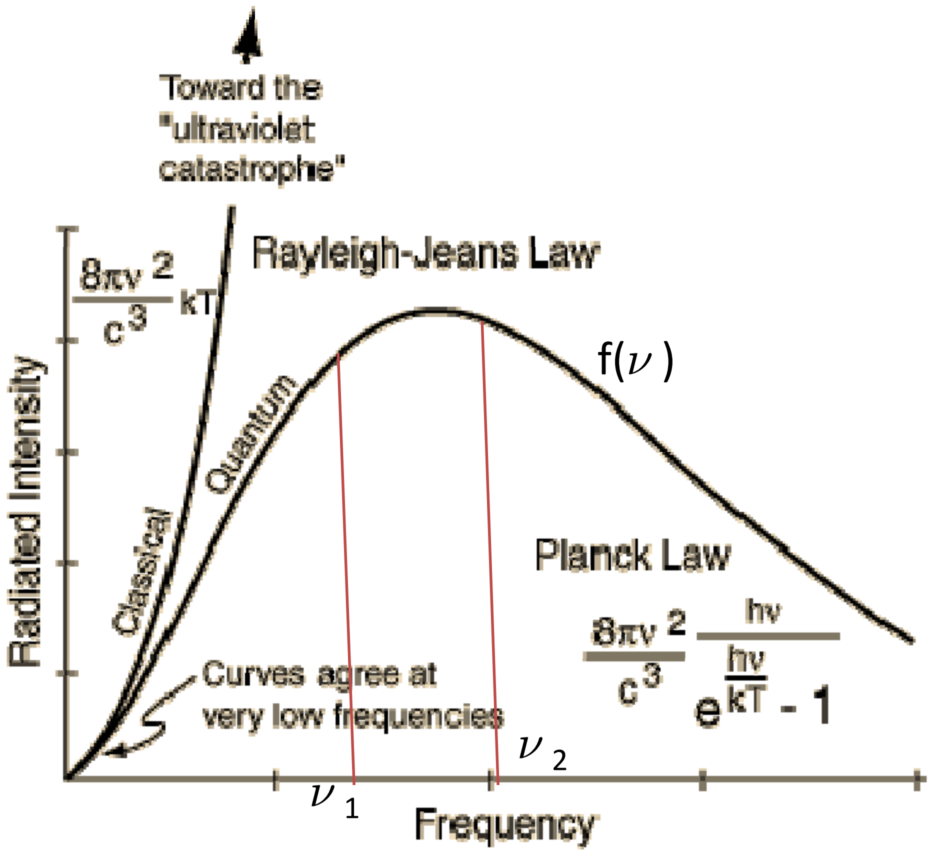
Conclusion for numerical simulations: # of grid points in space-time is *at least* $(Re)^{11/4}$. In reality, for numerical simulation one should probably take the time scale $\tau' \sim \lambda/U \sim R/U\ (Re)^{-3/4}$, (obtained from λ and *Taylor's hypothesis of frozen turbulence)*, which leads to $\sim (Re)^3$ space-time grid points.

4. Kolmogorov spectrum

We have determined the highest significant space and time frequencies in the flows. What about the whole spectrum of the velocity field, how much energy is in which frequency, both spatial and temporal?

Analogy:

Recall the classical "black body radiation" spectrum in statistical physics:



$$\int_{\nu_1}^{\nu_2} f(\nu)\, dv$$

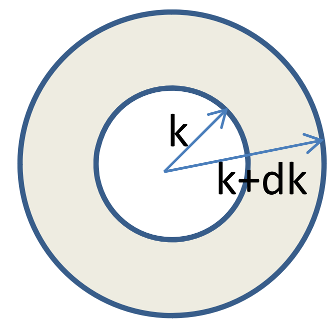
Energy per unit time radiated in frequencies between ν_1 and ν_2

Ω – a region of fluid of unit mass

E(k) dk - energy contained in the velocity field u in Ω in (spatial) frequencies with magnitudes between k and k+dk (??? – see below)

If u was supported in Ω we would write

$$
\begin{aligned}
u(x) &= (2\pi)^{-3}\int_{R^3} \hat{u}(\xi) e^{i\xi x}\, d\xi \\
\int |u|^2\, dx &= (2\pi)^{-3}\int |\hat{u}(\xi)|^2\, d\xi
\end{aligned}
$$

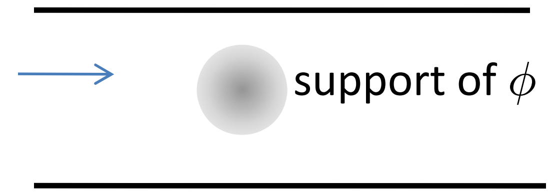


cartoon picture in the frequency space

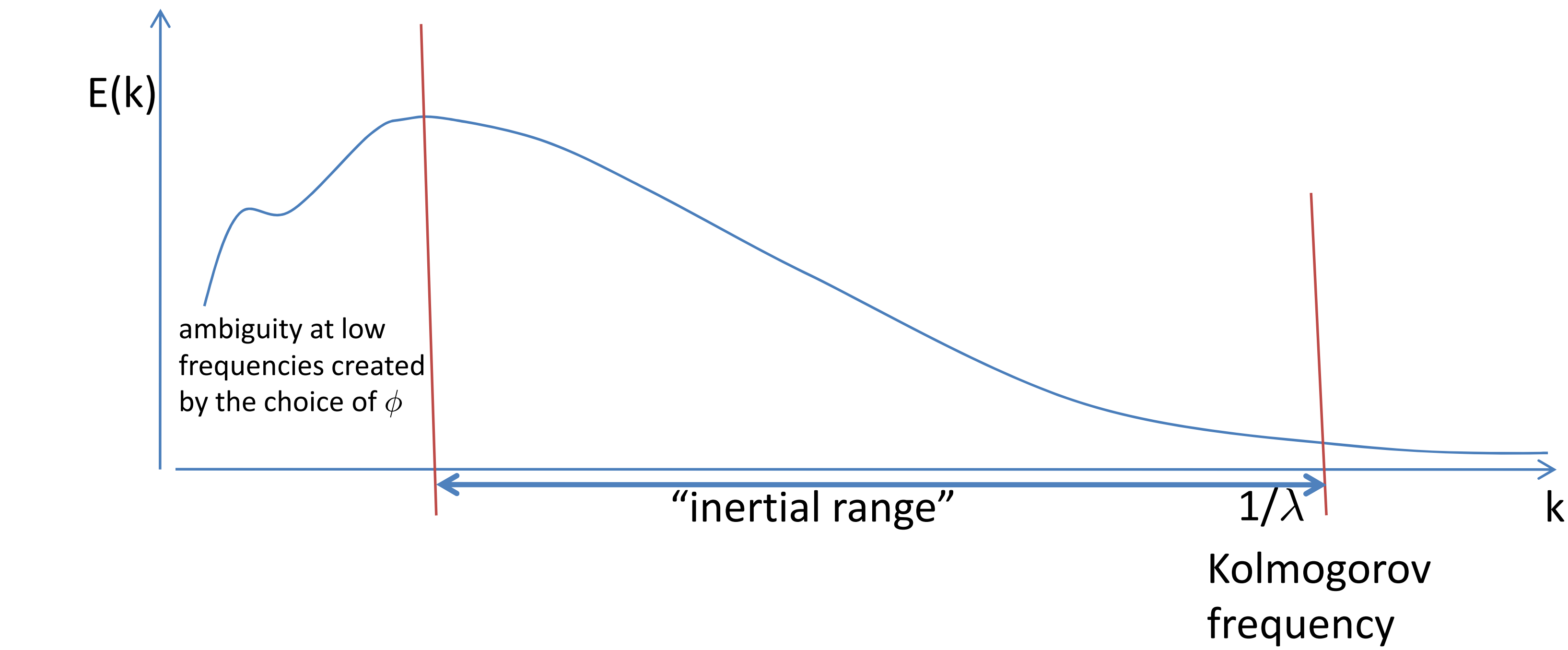
and

$$
(2\pi)^{-3}\int_{R^3} |\hat{u}(\xi)|^2\, d\xi = (2\pi)^{-3}\int_0^\infty \int_{S^2} |\hat{u}(k\eta)|^2 k^2\, d\eta\, dk = \int_0^\infty E(k)\, dk
$$

In general we replace Ω by a smooth cut-off function ϕ(x) with $\int \varrho\phi^{2}=1$ and replace u by ϕu in the above formulae. Remember that we really consider some ensemble averages, so this procedure is OK.



$$\int_{R^3} |\phi u|^2 \, dx = (2\pi)^{-3} \int_{R^3} |\widehat{\phi u}|^2 \, d\xi = \int_0^\infty k^2 dk \int_{S^2} |\widehat{u\phi}|^2 (k\eta) \, d\eta = \int_0^\infty E(k) \, dk$$



Kolmogorov: in the inertial range, E(k) should depend only on the "macroscopic quantities" ϵ, k, and ϱ.

dimensions: dim [E(k) dk] = dim [energy]/ [mass]= $[\text{length}]^2$ $[\text{time}]^{-2}$

dim dk = dim k = $[\text{length}]^{-1}$

dim E = $[\text{length}]^3$ $[\text{time}]^{-2}$

dim [ϵ] = $[\text{length}]^2$ $[\text{time}]^{-3}$

The only possible formula for E:

$$\mathbf{E(k) = const.\ (\epsilon)^{2/3}\ k^{-5/3}}$$ Kolmogorov-Obukhov 5/3 law

"spatial energy spectrum of turbulent velocity field"

Temporal spectra:

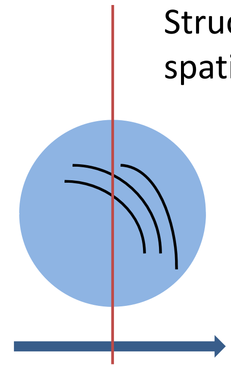
a) In a frame at rest with respect to space
   for a fixed x consider the functions t $\to$ u(x,t) = u(t) and its frequencies
   conceptually - Fourier decomposition of t $\to$ u(t) on a unit time interval
   (or an interval of a fixed length – we are working modulo multiplicative constants) .
   ϕ –compactly supported in the unit interval, with $\int |\phi(t)|^2$ dt=1.

$$\begin{aligned} \phi(t)u(t) &= (2\pi)^{-1}\int \widehat{\phi u}(\omega)e^{i\omega t}d\omega \\ \int |\phi u|^2\, dt &= (2\pi)^{-1}\int |\widehat{\phi u}|^2\, d\omega = \int E(\omega)d\omega \end{aligned}$$

If the “macroscopic” velocity U near x does not vanish , we have three natural quantities on which E(ω) can depend: ϵ, ω, U.

Taylor's "frozen turbulence" hypothesis

Time frequencies behave as if they were generated by watching at a fixed point a "frozen field" passing by at speed U



Temporal frequency of crossing a fixed point in space: ω = U k cos α

α angle between the vectors **U** and **k**.

If the space structures satisfy a k^{γ} power-law, the temporal frequencies also satisfy an ω^{γ} power law.

Dimensions: dim ω = $[\text{time}]^{-1}$

dim ($E(\omega)\, d\omega$) = dim u^2 = $[\text{length}]^2/[\text{time}]^2$

dim $E(\omega)$ = $[\text{length}]^2/[\text{time}]$

Conclusion: $\mathbf{E(\omega) = const.\ (U\epsilon)^{2/3}\omega^{-5/3}}$

b) Temporal frequencies in the frame of the moving particle

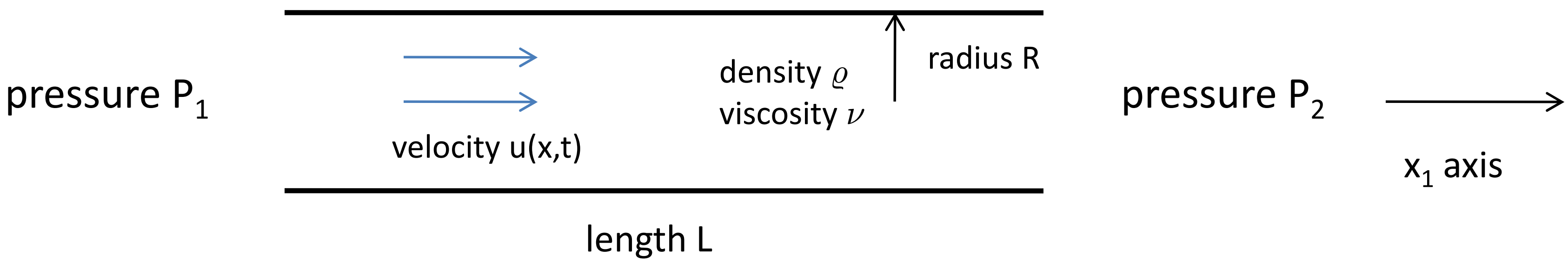
Now the only relevant quantities are ϵ, ω, E(ω)
same dimensions as before, (although the meaning of E(ω) and ω is modified)

Formula for E(ω) based on dimensional analysis

$$\mathbf{E(\omega) = const.}\ \epsilon\, \omega^{-2}$$

Reynolds Stress, Reynolds equations, the closure problem

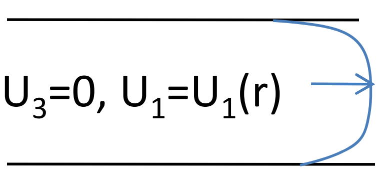
Consider again the pipe flow:



$$u_t - \nu \Delta u + u\nabla u + \nabla p/\varrho = 0$$
$$\operatorname{div} u = 0$$
$$\text{boundary condition} \qquad u = 0$$

Note the change of notation

Write the solutions u as $u=U+v$, where U is the “average” and v is the “fluctuations”.
Often notation $U=\langle u\rangle$ is used. In the simple situation of pipe flow we can think of time averages.

$U_2=U_3=0,\ U_1=U_1(r)$ 

Try to get an equation for $U_1(r)$ by averaging Navier-Stokes

Averaging the equations:

We write Navier-Stokes as

$$u_t - \nu\Delta u + \mathrm{div}(u \otimes u) + \nabla p = 0$$

Write $u(x,t) = U(x) + v(x,t)$
$p(x,t) = P(x) + q(x,t)$

note that <U⊗v> = 0

Pressure
average

$$-\nu\Delta U + \mathrm{div}\ (U \otimes U + < v \otimes v >) + \nabla P = 0$$

$$< v_i v_j >= R_{ij}$$

$$-\nu\Delta U + \mathrm{div}\ (U \otimes U + R) + \nabla P = 0$$

R is unknown

Try to write equations for $u_i\, u_j$ which would follow from Navier-Stokes for u and average these equations.

Obtain equations for R_{ij} which however contain averages of cubic quantities $\langle u_i\, u_j\, u_k \rangle = R_{ijk}$ (among other things).

Equations for R_{ijk} ? need (among other things) the fourth order moments $\langle u_i\, u_j\, u_k\, u_l \rangle$

We can try to continue, but new unknowns keep emerging and the system never closes - this is the closure problem.

Exercise: calculation in the channel flow (or the pipe flow) as $\nu \to 0$.