Kolmogorov’s 1941 Theory

Introduction
Richardson’s idea of the independence of the macroscopic flow features on viscosity

cartoon picture (in reality it is important that the flow is 3D)

larger whirls  
smaller whirls

larger viscosity

extinction due to viscosity at some length scale

smaller viscosity

extinction due to viscosity at smaller length scale

The net macroscopic effect remains (approximately) the same
Let’s apply it to pipe flows:

\[-\nu \Delta u + u \nabla u + \nabla p/\rho = 0\]
\[\text{div } u = 0\]
boundary condition \[u = 0\]

\[\text{pressure } P_1 \quad \text{mean velocity } U \text{ (unknown)} \quad \text{viscosity } \nu \quad \text{radius } R \quad \text{length } L \quad \text{pressure } P_2 \quad x_1 \text{ axis}\]

Problem: How much fluid flows through the pipe per unit of time?
Or: what is the mean velocity of fluid \(U\) in the pipe (defined by \(\pi R^2 U = \text{amount of fluid per unit time}\))?

Attempt #1: Explicit solution of Navier-Stokes!

\[u_1 = 2U \left(1 - \left(x_2^2 + x_3^2\right)/R\right), \quad u_2 = u_3 = 0\]
\[p = 8\rho \nu U x_1 / R^2 + \text{const.}\]
\[U = (P_1 - P_2) R^2 / 8\rho \nu L\]

“Poiseuille's flow”

Only works for real flows when \(\text{Re} = UR/\nu\)
is below a few thousand; recall \(\nu = 10^{-6}\) for water
Richardson’s philosophy: \textit{U should be independent of }\nu\textit{ once }Re\textit{ is large}
\textit{(and the flow becomes turbulent)}

Available quantities, assuming translational symmetry in }x_1:\textit{

\[ P’ = (P_1 - P_2) / L \text{ (pressure drop per unit length), } R, \rho, \text{ (but not } \nu \text{!)} \]

\textbf{Task: express }U\textbf{ from these quantities in a way which is independent of the choice of units of measurement.}

dimensions:
\[ [P’] = [\text{force}][\text{length}]^{-3} = [\text{mass}][\text{length}]^{-2}[\text{time}]^{-2} \]
\[ [R] = [\text{length}] \]
\[ [\rho] = [\text{mass}][\text{length}]^{-3} \]
\[ [U] = [\text{length}][\text{time}]^{-1} \]

\textit{The only possibility: }U = \textit{const. } (P’/\rho)^{1/2} \ R^{1/2} \]

In practice the formula works quite well – a remarkable fact given the rough derivation! Corrections to the formula: \textit{c=c(Re, roughness coefficient of the wall), but it does become almost constant for large Re and a given wall roughness. It seems to be absolutely hopeless to try to derive this from Navier-Stokes.}
Another way to look at the derivation

Pressure = L so that pressure per unit length = 1

Mean velocity \( u \) (unknown)

Density = 1

Viscosity \( \nu' \)

Radius = 1

Pressure = 0

Length \( L \)

In our set-up \( \text{Re}' \) would differ by a constant factor from the standard \( \text{Re}=RU/\nu \)

Normalize by suitable choice of units of length, time, mass; then \( u \) should be independent of \( \nu' \) (which plays the role of \( 1/\text{Re}' \))

Go back to the original units and recover \( U = \text{const.} \left( \frac{P'}{\rho} \right)^{1/2} R^{1/2} \)

with \( \text{const.} = u \)
The same argument in a slightly different language:

1. Assume the formula $U = f(P', R, \varrho)$ is independent of the choice of units.

2. Change units:

   - [old length] = $\lambda$ [new length]
   - [old time] = $\tau$ [new time]
   - [old mass] = $\mu$ [new mass]

3. Then $[P'] = [\text{force}] / [\text{length}]^3 = [\text{mass}] [\text{length}]^{-2} [\text{time}]^{-2}$

   and hence $[\text{old } P'] = \mu \lambda^{-2} \tau^{-2}$ [new $P'$]

   similarly $[\text{old } R] = \lambda$ [new R]

   $[\text{old } \varrho] = \mu \lambda^{-3}$ [new $\varrho$]

   and $[\text{old } U] = \lambda \tau^{-1}$ [new $U$]

4. The formula $U = f(P', R, \varrho)$ is the same in the old and the new quantities
   therefore $f(\mu \lambda^{-2} \tau^{-2} P', \lambda R, \mu \lambda^{-3} \varrho) = \lambda \tau^{-1} f(P', R, \varrho)$

   which is the same as

   $f(\alpha P', \beta R, \gamma \varrho) = \alpha^{1/2} \beta^{1/2} \gamma^{-1/2} f(P', R, \varrho)$

   or

   $U = f(P', R, \varrho) = P'^{1/2} \varrho^{1/2} R^{1/2} f(1,1,1)$
In terms of Navier-Stokes, the “normalization” is related to the **scaling symmetry**:

(which also plays a crucial role in the regularity theory)

If \( u(x,t), p(x,t) \) solve Navier Stokes with a given viscosity \( \nu \)

for any \( \lambda > 0 \)

\( \lambda u(\lambda x, \lambda^2 t), \lambda^2 p(\lambda x, \lambda^2 t) \)

also solve Navier-Stokes (with the same viscosity)

**Not good for scale models** (as can also be seen by using the Reynolds number):

1:10 scale model, (corresponding to \( \lambda = 10 \))

**velocity should be 10 \( U \)**

(can be hard to achieve in practice in some cases)

also, our **camera should run 100 times faster**

In reality, the situation is not so bad, due to the Richardson effect.
Following Kolmogorov (1941), we introduce

\[ \epsilon = \text{rate of energy dissipation in the fluid per unit mass} \]

It is not the instantaneous dissipation (which would depend on time), but a suitable average (either over time, or a statistical ensemble). It can depend on \( x \).

For example, in the pipe flows we can imagine the following:

\[ \epsilon = \epsilon(r) \text{ with } r = \text{distance to pipe axis} \]

In fact, we can imagine that \( \epsilon(r) \) be nearly constant in \( r \), except possibly near the boundaries, when \( r \sim R \). In practice this is not a bad assumption. (We will look at the possible \( r \)-dependence later.)
The variable $\varepsilon$ should be considered as a “macroscopic quantity”, similar to $U$, $P'$, etc., in contrast to the “microscopic field” $u(x,t)$. Typically we do not need all the microscopic details of $u(x,t)$.

According to Kolmogorov, $\varepsilon$, together with the density $\rho$ are the basic “macroscopic” quantities characterizing the flow. The key point now is

**The main assumption: $\varepsilon$ does not depend on $\nu$**

**Typical considerations**

1. **Dependence of $\varepsilon$ on $U$ and $R$ assuming $\varepsilon$ is constant in $r$.**

   - Dimension of $\varepsilon$: $[\text{mass}][\text{velocity}]^2 [\text{time}]^{-1} [\text{mass}]^{-1} = [\text{length}]^2 [\text{time}]^{-3}$
   - Dimension of $R$: $[\text{length}]$
   - Dimension of $U$: $[\text{length}][\text{time}]^{-1}$

   If $\varepsilon = \varepsilon(U,R)$, independently of the choice of units, we must have $\varepsilon = \text{const. } U^3/R$
2. Kolmogorov Length:

What are the dimension $\lambda$ of the smallest whirls in the flow?
Or: what is the space resolution $\lambda$ we need for a full numerical simulation?
Or: what is the smallest length-scale $\lambda$ in the flow?

Kolmogorov: The only relevant quantities are $\epsilon$, $\rho$, and $\nu$

dimensions: $[\epsilon] = [\text{length}]^2 [\text{time}]^{-3}$, $[\nu] = [\text{length}]^2 [\text{time}]^{-1}$, $[\rho] = [\text{mass}][\text{length}]^{-3}$

$[\lambda] = [\text{length}]$

The only possible formula for $\lambda$ (assuming it exists):

$$\lambda = \text{const.} \, \epsilon^{-1/4} \, \nu^{3/4}$$ - “Kolmogorov length”

Recalling $\epsilon = \text{const.} \, U^3/ R$ and $\text{Re} = UR/\nu$, we obtain

$$\lambda = \text{const.} \, R / \text{Re}^{3/4}$$  (for typical garden hose flows $\text{Re} \sim 10^5 – 10^6$)
3. Kolmogorov Time:

What are the smallest time-scales $\tau$ (or the highest time frequencies $f = 1/\tau$)?

Relevant variables: $\epsilon$, $\nu$, $Q$

Note that we do not list the macroscopic velocity $U$. Therefore we are talking about local time-scales, as observed in the system of a moving particle.

Exercise: based on dimensional analysis, show that the only possible formula is

$$\tau = \text{const.} \, \epsilon^{-1/2} \, \nu^{1/2}$$

In terms of Reynolds number

$$\tau = \text{const.} \, (R/U) \, \text{Re}^{1/2}$$

of order $10^5$ sec in a typical garden hose flow.

Conclusion for numerical simulations: # of grid points in space-time is at least $(\text{Re})^{11/4}$. In reality, for numerical simulation one should probably take the time scale $\tau' \sim \lambda/U \sim R/U \, (\text{Re})^{-3/4}$, (obtained from $\lambda$ and Taylor’s hypothesis of frozen turbulence), which leads to $\sim (\text{Re})^3$ space-time grid points.
4. Kolmogorov spectrum

We have determined the highest significant space and time frequencies in the flows. What about the whole spectrum of the velocity field, how much energy is in which frequency, both spatial and temporal?

Analogy:
Recall the classical “black body radiation” spectrum in statistical physics:

\[\int_{\nu_1}^{\nu_2} f(\nu) \, d\nu\]

Energy per unit time radiated in frequencies between \(\nu_1\) and \(\nu_2\)
$\Omega$ – a region of fluid of unit mass

$E(k) \, dk$ - energy contained in the velocity field $u$ in $\Omega$ in (spatial) frequencies with magnitudes between $k$ and $k+dk$ (??? – see below)

If $u$ was supported in $\Omega$ we would write

$$u(x) = (2\pi)^{-3} \int_{\mathbb{R}^3} \hat{u}(\xi) e^{i\xi x} \, d\xi$$

$$\int |u|^2 \, dx = (2\pi)^{-3} \int |\hat{u}(\xi)|^2 \, d\xi$$

and

$$(2\pi)^{-3} \int_{\mathbb{R}^3} |\hat{u}(\xi)|^2 \, d\xi = (2\pi)^{-3} \int_0^\infty \int_{S^2} |\hat{u}(k\eta)|^2 k^2 \, d\eta \, dk = \int_0^\infty E(k) \, dk$$
In general we replace $\Omega$ by a smooth cut-off function $\phi(x)$ with $\int \phi^2 = 1$ and replace $u$ by $\phi u$ in the above formulae. Remember that we really consider some ensemble averages, so this procedure is OK.

$$
\int_{\mathbb{R}^3} |\phi u|^2 \, dx = (2\pi)^{-3} \int_{\mathbb{R}^3} |\hat{\phi} u|^2 \, d\xi = \int_0^\infty k^2 \, dk \int_{S^2} |\hat{u} \phi|^2 (k\eta) \, d\eta = \int_0^\infty E(k) \, dk
$$

![Graph showing the energy spectrum $E(k)$ with the "inertial range" and 1/$\lambda$ marked.]
Kolmogorov: in the inertial range, $E(k)$ should depend only on the "macroscopic quantities" $\epsilon$, $k$, and $\rho$.

**Dimensions:**

$$\text{dim } [E(k) \, dk] = \text{dim } [\text{energy}] / [\text{mass}] = [\text{length}]^2 [\text{time}]^{-2}$$

$$\text{dim } dk = \text{dim } k = [\text{length}]^{-1}$$

$$\text{dim } E = [\text{length}]^3 [\text{time}]^{-2}$$

$$\text{dim } [\epsilon] = [\text{length}]^2 [\text{time}]^{-3}$$

The only possible formula for $E$:

$$E(k) = \text{const. } (\epsilon)^{2/3} k^{-5/3} \quad \text{Kolmogorov-Obukhov 5/3 law}$$

"spatial energy spectrum of turbulent velocity field"
Temporal spectra:

a) In a frame at rest with respect to space for a fixed $x$ consider the functions $t \rightarrow u(x,t) = u(t)$ and its frequencies conceptually - Fourier decomposition of $t \rightarrow u(t)$ on a unit time interval (or an interval of a fixed length – we are working modulo multiplicative constants).

For a compactly supported in the unit interval, with $\int |\phi(t)|^2 \, dt = 1$.

\[
\phi(t)u(t) = (2\pi)^{-1} \int \hat{\phi}(\omega) e^{i\omega t} \, d\omega
\]

\[
\int |\phi u|^2 \, dt = (2\pi)^{-1} \int |\hat{\phi} u|^2 \, d\omega = \int E(\omega) \, d\omega
\]

If the “macroscopic” velocity $U$ near $x$ does not vanish, we have three natural quantities on which $E(\omega)$ can depend: $\epsilon$, $\omega$, $U$. 
Taylor’s “frozen turbulence” hypothesis

Time frequencies behave as if they were generated by watching at a fixed point a “frozen field” passing by at speed $U$

If the space structures satisfy a $k^γ$ power-law, the temporal frequencies also satisfy an $ω^γ$ power law.

Dimensions:

\[
\dim ω = [\text{time}]^{-1} \\
\dim (E(ω) dω) = \dim u^2 = [\text{length}]^2/[\text{time}]^2 \\
\dim E(ω) = [\text{length}]^2/[\text{time}]
\]

Conclusion: $E(ω) = \text{const.} \ (Uε)^{2/3} ω^{-5/3}$
b) Temporal frequencies in the frame of the moving particle

Now the only relevant quantities are $\epsilon, \omega, E(\omega)$
same dimensions as before, (although the meaning of $E(\omega)$ and $\omega$ is modified)

Formula for $E(\omega)$ based on dimensional analysis

$$E(\omega) = \text{const.} \, \epsilon \, \omega^{-2}$$
Reynolds Stress, Reynolds equations, the closure problem

Consider again the pipe flow:

\[ u_t - \nu \Delta u + u \nabla u + \nabla p/\rho = 0 \]
\[ \text{div } u = 0 \]
\[ \text{boundary condition } u = 0 \]

Write the solutions \( u \) as \( u = U + v \), where \( U \) is the “average” and \( v \) is the “fluctuations”. Often notation \( U = \langle u \rangle \) is used. In the simple situation of pipe flow we can think of time averages.

\[ U_2 = U_3 = 0, \quad U_1 = U_1(r) \]

Try to get an equation for \( U_1(r) \) by averaging Navier-Stokes
Averaging the equations:

We write Navier-Stokes as

$$u_t - \nu \Delta u + \text{div}(u \otimes u) + \nabla p = 0$$

Write $$u(x, t) = U(x) + v(x, t)$$

$$p(x, t) = P(x) + q(x, t)$$

note that $$\langle U \otimes v \rangle = 0$$

$$-\nu \Delta U + \text{div} \ (U \otimes U + \langle v \otimes v \rangle) + \nabla P = 0$$

$$\langle v_i v_j \rangle = R_{ij}$$

$$-\nu \Delta U + \text{div} \ (U \otimes U + R) + \nabla P = 0$$

Pressure average

Pressure is unknown
Try to write equations for $u_i u_j$ which would follow from Navier-Stokes for $u$ and average these equations.

Obtain equations for $R_{ij}$ which however contain averages of cubic quantities $<u_i u_j u_k> = R_{ijk}$ (among other things).

Equations for $R_{ijk}$ need (among other things) the fourth order moments $<u_i u_j u_k u_l>$

We can try to continue, but new unknowns keep emerging and the system never closes - this is the **closure problem**.
Exercise: calculation in the channel flow (or the pipe flow) as $\nu \to 0$. 