Equilibrium Pricing in Incomplete Markets - The Continuous Time Model -

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PIMS, July 2008

Outline

- Reminder: The multi-period model.
- The continuous time model.
- Equilibrium dynamics and BSDEs.
- Back to the discrete time model.

Literature: H, Pirvu & dos Reis "On Securitization, Market Completion and Equilibrium Risk Transfer", WP, (2008).

The model in discrete time

- We considered a dynamic incomplete market model with:
  - a finite set $\mathbb{A}$ of agents endowed with H^a ($a \in \mathbb{A}$)
  - a finite sample space $(\Omega, \mathscr{F}, \mathbb{P})$
- At time $t = 1, ..., T$ the agents maximize preference functionals

$$U_t^a : L(\mathscr{F}_T) \to L(\mathscr{F}_t)$$

that are concave, normalized, monotone, translation invariant,

$$U_t^a(X + Z) = U_t^a(X) + Z \quad \text{for all} \quad Z \in L(\mathscr{F}_t),$$

and time consistent, i.e.,

$$U_t^a(X) = U_t^a\left(U_{t+1}^a(X)\right) \quad \text{for all} \quad X \in \mathscr{F}_{t+1}.$$

- The illiquid asset paid dividends so its terminal value was given.

THE ILLIQUID ASSET WAS PRICED IN EQUILIBRIUM.

The Random Walk Framework

- The agents' preferences followed the backward dynamics:

$$U^a_{t+h} - U^a_h = f^a_t(Z^a_{t+h}) + Z^a_{t+h} \cdot \Delta b_{t+h}, \quad U^a_T = H^a.$$

- From this we concluded that

$$\begin{aligned} R_{t+h} - R_h &= Z^R_{t+h} \cdot \nabla f_t(0) + Z^R_{t+h} \cdot \Delta b_{t+h}, \qquad R_T = H \\ V^a_{t+h} - V^a_h &= g^a(Z^R_{t+h}, Z^a_{t+h}) + Z^a_{t+h} \cdot \Delta b_{t+h}, \quad V^a_T = V^a \end{aligned}$$

where

$$Z^R_{t+h} = \pi_t(R_{t+h}) \quad \text{and} \quad Z^a_{t+h} = \pi_t(V^a_{t+h}).$$

What happens if the time between two trading periods tends to zero?

Equilibrium Pricing in Continuous Time

- The agents' incomes are exposed to financial and non-financial risk:

$$W^a = H^a(S_T, R_T^a) \quad \text{where} \quad R_T^a \sim Q(R_T; \cdot).$$

- Individual risks R_T^a originate from a common risk factor R_T.

- Preferences are described dynamic translation invariant, ... preference functionals.

- A third party (insurance company) issues a derivative on R_T:

$$B_T = H(R_T).$$

- Exchange of risk exposure through trading the derivative.

The derivative in fixed supply and will be priced by an equilibrium approach.

The microeconomic setup

- The agents are exposed to financial and non-financial risk factors:

– Logarithmic Asset prices follow a diffusion process:

$$dS_t = \theta^S(R_t)dt + dW_t^S.$$

– The external risk process follows a Brownian motion with drift:

$$dR_t = \mu dt + dW_t^R.$$

- The agents have bounded random incomes of the form:

$$H^a = h^a(S_T, R_T + W_T^a) + \int_0^T \varphi_s^a(S_s, R_s + W_s^a)ds.$$

where all the Brownian motions $W^S, W^R, (W^a)$ are independent.

The market is incomplete even after the new asset is introduced.

The optimization problem

- Their risk preferences are described by backward stochastic differential equations (BSDEs):

$$-Y_t^a = H^a - \int_t^T g^a(s, X_s, Z_s^a)ds - \int_t^T Z_s^a \cdot dW_s$$

where $X = (S, R, (W^a)_{a\in\mathbb{A}})$ is the forward process.

- One can show (key!) that all of the entries of the vector

$$Z_t^a = \left(Z_t^S, Z_t^R, (Z_t^b)_{b\in\mathbb{A}}\right)$$

are zero except Z^S, Z^R, Z^a; these refer to the agents' risk exposure.

- We consider the case of entropic utility functions:

$$g^a(t,x,z) = \frac{1}{2\gamma_a}\|z\|^2 \qquad Y_t^a = \frac{1}{\gamma_a}\log\mathbb{E}[-e^{-\gamma_a H^a}|\mathscr{F}_t]$$

All BSDEs are assumed to satisfy a comparison principle.

The Derivative

- There is an insurance company or investment bank ...
  - ... that holds a portfolio of climate sensitive securities, and ...
  - ... issues a bond whose payoff B depends on the portfolio risk:

$$B = h(R_T) + \int_0^T \varphi_s(R_s)ds \quad \text{or} \quad B = H(R_T).$$

- It is in unit net supply and priced by an equilibrium condition:

$$B_t = \mathbb{E}_{\mathbb{P}^*}[B|\mathscr{F}_t] \quad \text{w.r.t an endogenous pricing measure } \mathbb{P}^*.$$

- We state conditions that guarantee that an equilibrium pricing measure exists.

We focus on RISK TRANSFER *rather than* RISK SHARING.

Pricing Schemes

• The derivative is priced in a market environment; hence by a linear pricing scheme

$$I : L^2(\mathbb{P}) \rightarrow \mathbb{R}_+.$$

• Each such scheme can be identified with a measure $\mathbb{Q} \approx \mathbb{P}$.

• We assume that the agents have no impact on stock prices so the restriction on $\mathscr{F}^S$ is given by the price of financial risk θ^S.

• Notice that the pricing rule is linear for the agents, not for the insurer.

THE MARKET PRICE OF EXTERNAL RISK WILL BE ENDOGENOUS.

Pricing Schemes

- The set of all possible pricing rules is given by

$$\mathscr{P} = \{\mathbb{Q} \approx \mathbb{P} \text{ and } S \text{ is a } \mathbb{Q}\text{-martingale}\}.$$

- For each $\mathbb{Q} \in \mathscr{P}$ the density is a uniformly integrable martingale:

$$Z_t = \exp\left(-\int_0^t \begin{pmatrix} \theta^S \\ \theta_s^R \end{pmatrix} d \begin{pmatrix} W_s^S \\ W_s^R \end{pmatrix} - \frac{1}{2}\int_0^t |\theta_s|^2 ds \right)$$

- The set of all pricing linear rules can be identified with the set of

market prices of external risk $\theta^R = (\theta_s^R)$

such that (Z_t) defined by the above equation is an u.i. martingale.

THE SET OF PRICING RULES IS IDENTIFIED BY THE MARKET PRICES OF EXTERNAL RISK.

Risk Sharing vs Risk Transfer

- In a model of risk sharing the the bond pays no dividends.

- Exchange of risk exposures takes place through a fictitious asset:

$$d\bar{B}_t^{\theta} = \theta_t^R dt + dW^R$$

with a given volatility process that can be normalized.

- In a model of risk transfer exchange of risk exposures takes place through market prices:

$$dB_t^{\theta} = \kappa_t^R(\theta_t^R dt + dW^R) + ...dt + ...dW_t^S.$$

with an endogenous volatility process that cannot be normalized!

We characterize the equilibrium market price of risk and represent κ^R.

Utility Optimization and Market Clearing

- There is no a-priori reason that the bond "adds somethings".

- Given a candidate θ^R for the market price of external risk:

– ... let us assume the market is complete (we verify this later), ...
– ... and introduce a pricing measure $\mathbb{P}^\theta$ along with ...
– ... the corresponding bond price process (B_t^θ), ...
– ... and solve the agents' optimization problems.

- For a given admissible trading strategy π in both markets:

$$-dY_t^a(\pi) = g^a(t, X_t, Z_t)dt - Z_t dW_t$$

with terminal condition

$$Y_T^a(\pi) = -H^a - V_T^{a,\theta}(\pi).$$

- The agent's goal is the minimize $Y_0^a(\pi)$.

We first solve this problem for a given pricing measure.

Optimal Trading Strategies and Equilibrium

- Let (Y^a, Z^a) be the unique solution of the agent's utility BSDE.
- For a given measure $\mathbb{P}^\theta$ the optimal strategy π^a is of the form:

$$\pi_t^a = (\pi_t^{a,S}, \pi_t^{a,B}) = G^a(t, Z_t^a, \theta_t^S, \kappa_t^R).$$

- We need to satisfy the equilibrium condition:

$$\sum_a \pi_t^{a,B} = 1.$$

- General equilibrium theory in a complete market environment:

competitive equilibria $\leftrightarrow$ representative agent equilibria.

- Due to the the specific structure of idiosyncratic risk exposures:

Analysis can be reduced to a representative agent economy.

WE CAN DESCRIBE EQUILIBRIUM PRICES BY A SINGLE BSDE.

The Representative Agent

- Assume that only two agent are active in the market: $\mathbb{A} = \{a, b\}$.

- The representative agent minimizes aggregate risk: the BSDE is

$$-dY_t^{ab} = g^{ab}(t, X_t, Z_t)dt - Z_t dW_t$$

with the terminal condition

$$Y_T^{ab}(\pi) = -H^a - H^b - H - V_T^{ab,\theta}(\pi)$$

where the driver $g^{a,b}$ is defined by the inf-convolution:

$$g^{ab}(t, z) = \inf_x \{g^a(t, z - x) + g^b(t, x)\} = \frac{1}{2\gamma}\|z\|^2.$$

- Under some assumptions the agent's minimization problem has a solution for a given pricing measure.

We choose θ^R such that $\pi_t^{ab,B} \equiv 0$.

The Equilibrium Market Price of External Risk

Theorem: Assume that the derivative's payoff is strictly monotone in the external risk process (R_t) and consider the quadratic BSDE

$$dY_t = -Z_t dW_t + \frac{1}{2}\left[-(Z_t^R)^2 + (\theta^S)^2 - 2\theta^S Z_t^S - \sum_{a \in \mathbb{A}} (Z_t^a)^2\right] dt$$

with terminal condition $Y_T = -H^{ab}$ where

$$Z = \left(Z^S, Z^R, (Z^a)_{a \in \mathbb{A}}\right).$$

Then Z^R is an equilibrium market price of external risk. Under additional assumptions each Z^a is bounded.

WHAT DOES ALL THIS HAVE TO DO WITH DISCRETE TIME MODELS?

The Equilibrium Market Price of External Risk

Theorem: If the preceding BSDE has a unique solution such that Z^a is bounded, then the solution can be approximated by equation in discrete time of the form:

$$\bar{Y}_t = \mathbb{E}[\bar{Y}_{t+h}|\mathscr{F}_t] + f_t(\bar{Z}_{t+h})$$

which is just the equilibrium dynamics corresponding to a discrete time model.

Corollary: The equilibrium dynamics of the discrete time model converge to an equilibrium dynamics of a continuous time model.

Notice: The convergence result requires an equilibrium in the continuous time!

Summary and Conclusion

- We considered discrete time model of general equilibrium pricing under translation invariant preferences.
- Equilibria exist if and only if a representative agent exists.
- In a random walk framework equilibria can be characterized in terms of a coupled system of backward equation.
- For a simple benchmark model: convergence of equilibrium dynamics to equilibria of a continuous time model
- Existence and differentiability if quadratic BSDEs.
- Applications of BSDEs to problems of cross hedging.