KK-Theory and Baum-Connes Conjecture

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Throughout this lecture, groups are topological groups (in particular, they are locally compact). All maps are topological (*i.e.*, continuous).

1 Preliminaries (Heath Emerson, June 14)

1.1 Some compact groups:

1. Circle $\mathbb{T}$
2. $U(n)$, $SU(n)$, $SO(n)$

3. $\mathbb{Z}_p$, p-adic integers.

1.2 A G-space is a locally compact space X equipped with a continuous map

$$\begin{array}{ccc} G \times X & \to & X \\ (g, x) & \mapsto & g \cdot x \end{array}$$

such that $g \cdot (g' \cdot x) = (gg') \cdot x$ and $e \cdot x = x$.

1.3 Example (Important example of a G-space) Let H be a closed subgroup of a compact group G. Then the homogeneous space $X := G/H$ is a compact Hausdorff space. G acts on X; $g \cdot (g_1 H) = gg_1 H$. And $X/G \simeq \mathrm{pt}$.

1.4 The isotropy subgroup (*i.e.*, stabilizer) of $x \in X$ is a closed subgroup of G.

Exercise 1 Let $X = \mathbb{C}P^n$. This is a G-space for $G = U(n+1)$. Check that

$$\mathbb{C}P^n \simeq U(n+1)/U(1) \times U(n)$$

where

$$U(1) \times U(n) \simeq \left\{ \left(\begin{array}{c|ccc} U(1) & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & U(n) & \\ 0 & & & \end{array} \right) \in G \right\}$$

1.5 Example Let X be a smooth compact manifold. Let $g : X \to X$ be a diffeomorphism. Then g is part of a compact group action iff g preserves some Riemannian metric on X. (Main fact used to prove this is that the isometry group of X with a fixed given metric $\langle \, , \rangle$ is compact.)

1.6 The category of G-spaces A map $\phi : X \to Y$ between G-spaces is a G-map if

$$\phi(g \cdot X) = g \cdot \phi(Y).$$

"equivariant." We may talk about isomorphic G-spaces.

1.7 Induction and restriction Let H be a closed subgroup of G. A G-space, restricting the action to H, gives an H-space. This is the restriction functor Res. The induction functor Ind is in the other direction; given an H-space Y, the induced G-space is $G \times_H Y := (G \times Y)/\sim$, where $(g, y) \sim (gh, h^{-1}y)$.

Exercise 2 If X is a G-space that admits a (surjective) G-map $\pi : X \to G/H$, then X is an induced space from H. (Hint: Consider the G-map

$$\begin{array}{rccc} \phi : & G \times_H Y & \to & G/H \\ & [g, y] & \mapsto & [g]. \end{array}$$

Take $Y = \phi^{-1}(eH)$.)

1.8 Lemma

$$\mathrm{Ind}(\mathrm{Res}\, X) \simeq X \times (G/H)$$

where G acts diagonally on $X \times (G/H)$.

Why is this true? Define

$$\begin{array}{rccc} \phi : & G \times_H X & \to & (G/H) \times X \\ & [g, x] & \mapsto & ([g], g \cdot x). \end{array}$$

Check that ϕ is a G-map that is a homeomorphism. □

1.9 Theorem (Palais) Let G be a Lie group, acting properly on a space X, let $x \in X$, and let $H = G_x$. Then there exists a G-invariant open neighbourhood U of x and a G-map

$$U \to G/H.$$

Thus U is induced from G_x.

Remark.

1. Thus, if G acts freely on X, then locally X looks like $G \times Y$ for some Y.
2. Assume that X is a smooth manifold with G acting smoothly. Consider the orbit $G \cdot x$ in X. Let N be the normal bundle. Then the fibre N_x of x is a G_x-space with a linear action. The proof of Palais' Theorem shows that we may take $U = G \times_H N_x$.

Exercise 3 Let $G = \prod_{n\in\mathbb{Z}} Z_2$. (This is not a Lie group.) Let $X = \prod_{n\in\mathbb{Z}} S^1$. Show that there is no G-invariant neighbourhood of $(1, 1, 1, \dots)$ which is induced from the isotropy subgroup of $(1, 1, 1, \dots)$.

Vector Bundles

1.10 Definition (Atiyah) A continuous family of (real or complex) vector spaces over X is a space E equipped with

1. a map $\pi : E \to X$,

2. and a finite-dimensional vector space structure on each fibre $E_x = \pi^{-1}(x)$, compatible with the topology of E (*i.e.*, the topology on E_x is the subspace topology induced from E).

A homomorphism of families E and E' over X is a map $\varphi : E \to E'$ s.t. $\pi_{E'} \circ \varphi = \pi_E$ and $\varphi\restriction_{E_x} : E_x \to E'_x$ is linear. φ is an isomorphism if φ is a homeomorphism.

1.11 Pullbacks If $f : X \to Y$ is a map, $\pi : E \to Y$ is a continuous family of vector spaces over Y, then

$$f^*(E) \overset{\text{def}}{=} \{ (x, v) \in X \times E \mid x \in X, v \in E_{f(x)} \}.$$

1.12 Example If $E \to X$ is a continuous family of vector spaces and A is a subspace of X, then for the inclusion $i : A \hookrightarrow X$, we have $i^*E = E\restriction_A$.

1.13 A continuous family of vectors spaces is **trivial** if it is isomorphic to $X \times \mathbb{C}^n$ (or $X \times \mathbb{R}^n$) for some n.

1.14 Definition A **vector bundle** over X is a continuous family of vector spaces such that, for all $x \in X$, there is a neighbourhood U of x s.t. $E\restriction_U$ is trivial.

1.15 Definition A G-equivariant vector bundle (G-bundle for short) is a G-space E over a G-space X such that $g \circ \pi_E = \pi_E \circ g$ and that $g\restriction_{E_x}$ is linear.

1.16 Example Let X be a G-space, and $\pi : G \to GL(V)$ a finite-dimensional representation. Then $X \times V$ with diagonal G-action is a G-equivariant vector bundle. Such a G-bundle is said to be trivial (*i.e.*, diagonal with a fixed representation on V).

2 Equivariant K-theory (Heath Emerson, June 15)

2.1 Example If G acts smoothly (by diffeomorphisms) on a compact manifold X, then differentiating the G-action gives TX the structure of a G-equivariant vector bundle.

Remark. Let X, Y be G-spaces. Isomorphism classes of G-equivariant vector bundles on X classify smooth G-equiv embeddings $X \to Y$ by mapping an embedding to its normal bundle.

2.2 Example If $H \leq G$ is a closed subgroup and $\pi : H \to GL(V)$ is a finite-dimensional representation of H, then $G \times_H V$ is a G-equivariant vector bundle over G/H. (The H-action on the fibre of eH is the original representation π.) This is called the "induced" vector bundle.

If π extends to a representation of G, then $G \times_H V \simeq (G/H) \times V$ is trivial.

2.3 Pullbacks Let X, Y be G-spaces. Let $\phi : X \to Y$ be a G-map. Let V be a G-equivariant vector bundle over Y. Then

$$\phi^*(V) = \{ (x, v) \mid v \in V_{\phi(x)} \}.$$

This is a G-equivariant vector bundle.

2.4 Homotopy invariance If ϕ_0 and $\phi_1, X \to Y$ are G-maps that are homotopic, then $\phi_0^*(V) \simeq \phi_1^*(V)$ for any G-equivariant vector bundle V over Y.

2.5 Operations on vector bundles Sums:

$$V_1 \oplus V_2 = \{ (v_1, v_2) \in V_1 \times V_2 \mid \pi_{V_1}(v_1) = \pi_{V_2}(v_2) \}.$$

Tensor products: $V_1 \otimes V_2$ has fibre, over x, $(V_1)_x \otimes (V_2)_x$.

2.6 Definition

$$\mathsf{Vect}_G(X) = \text{monoid of isomorphism classes of } G\text{-bundles over } X$$

2.7 Module structure Twisting by a representation: Given a G-bundle $V \to X$ and a representation $\pi : G \to GL(E)$, form a G-bundle over X, denoted $V \otimes E$, whose fibre over x is $V_x \otimes E$. So it is $V \otimes (X \times E)$ as tensor product of vector bundles.

2.8 Definition The G-equivariant K-theory, $K_G^0(X)$, of X is the Grothendieck group of the monoid $\mathsf{Vect}_G(X)$. It is a ring and a module over $R(G)$. Note that

$$R(G) = K_G^0(pt).$$

Remark.

1. If G is trivial, then $R(G) = \mathbb{Z}$, and $K_G^0(X)$ is only an abelian group.
2. If G acts trivially on X, then $K_G^*(X) \simeq K^*(X) \otimes_{\mathbb{Z}} R(G)$.

2 EQUIVARIANT K-THEORY (HEATH EMERSON, JUNE 15)

2.9 Example If $G = \mathbb{T}$, then

$$R(\mathbb{T}) = \mathbb{Z}[\hat{\mathbb{T}}] \simeq \mathbb{Z}[X, X^{-1}].$$

Given a space X with $\mathbb{T}$-action, $K^0_{\mathbb{T}}(X)$ is a module over $\mathbb{Z}[X, X^{-1}]$.

2.10 Theorem (Swan's Theorem) Let X be compact, G be trivial. Then

$$V \mapsto \Gamma(V), \text{space of sections}$$

induces an isomorphism of monoids

$$\mathrm{Vect}_G(X) \simeq \{\text{finitely generated projective modules over } C(X)\}.$$

The Murray-von-Neumann classes of projections in $\mathbb{M}_n(C(X))$ give the K-theory of $C(X)$. We conclude that

$$K^0(X) \simeq K_0(C(X)).$$

2.11 Noncompact spaces If X is non-compact,

$$K^0_G(X) = \ker(\varepsilon^* : K^0_G(X^+) \to K^0_G(\infty) = R(G))$$

where $\varepsilon : \infty \hookrightarrow X^+$.

2.12 Higher K-groups

$$K^{-n}_G(X) \overset{\text{def}}{=} K^0_G(X \times \mathbb{R}^n).$$

$n \in \mathbb{N}$, where G acts trivially on $\mathbb{R}^n$.

2.13 Theorem (Bott Periodicity) For complex vector bundles.

$$K^i_G(X) \simeq K^{i+2}_G(X).$$

In particular

$$K^0_G(\mathbb{R}^n) = \begin{cases} R(G), & n \textit{ even} \\ 0, & n \text{ odd}. \end{cases}$$

2.14 Long exact sequence If $Y \subseteq X$ is a closed G-invariant subset, then there is a long exact sequence

$$\begin{array}{ccccc} K^0_G(X \setminus Y) & \longrightarrow & K^0_G(X) & \longrightarrow & K^0_G(Y) \\ \uparrow & & & & \downarrow \\ K^1_G(Y) & \longleftarrow & K^1_G(X) & \longleftarrow & K^1_G(X \setminus Y) \end{array}$$

Remark. Wrong-way maps from open embeddings. $f : X \to Y$ induces $f^* : K_G^*(Y) \to K_G^*(X)$. If $U \subseteq X$ is an open, G-invariant subset, then there is a canonical G-map $X^+ \to U^+$ mapping points out of U to the point at infinity. This induces a map

$$K_G^*(U^+) \to K_G^*(X^+).$$

This gives

$$K_G^*(U) \to K_G^*(X).$$

Exercise 4 Let $G = \mathbb{Z}/2\mathbb{Z}$ act on S^1 by reflection. Calculate $K_G^*(S^1)$.

Exercise 5

$$K_G^i(G/H) \simeq \begin{cases} R(H), & i = 0, \\ 0, & otherwise. \end{cases}$$

Exercise 6 Let G act freely on X. Then $K_G^*(X) \simeq K^*(G\backslash X)$. Prove this for G a Lie group, using Palais' Theorem from last time.)

2.15 Example Let $G = \mathbb{T} \subseteq SU(2) \subseteq Homeo(\mathbb{C}P^1)$. If H is the dual H^{**} of the Hopf bundle

$$H^* = \{ (\ell, v) \mid v \in \ell \}$$

and if X is the class in $K_G^0(\mathbb{C}P^1)$ of the trivial G-bundle $\mathbb{C}P^1 \times \mathbb{C}$ with the standard representation of $\mathbb{T} \subseteq \mathbb{C}^\times$, then

$$K_G^1(\mathbb{C}P^1) = 0$$

and $K_G^0(\mathbb{C}P^1)$ is generated as a $\mathbb{Z}[X, X^{-1}]$-algebra by X and [H] (recall $\mathbb{Z}[X, X^{-1}] = R(\mathbb{T})$) with the relation

$$([H] - X)([H] - X^{-1}) = 0.$$

Moreover (exercise), $K_G^0(\mathbb{C}P^1)$ is a free $\mathbb{Z}[X, X^{-1}]$-module with generators 1 and [H].

1 always denotes the unit in $K_G^0(X)$, the class of $X \times \mathbb{C}$ with trivial G-action on $\mathbb{C}$.

3 Clifford Symbols (Heath Emerson, June 16)

3.1 Definition A G-bundle $V \to X$ is **subtrivial** if there is a G-bundle $V^\perp$ such that $V \oplus V^\perp$ is trivial.

3.2 Example Let $X = \mathbb{Z}$, $G = \mathbb{T}$. Suppose G acts trivially on X. Let $V = X \times \mathbb{C}$. Suppose $z \in \mathbb{T}$ acts on V by $z \cdot (n, \lambda) = (n, z^n \lambda)$ (the action changes fibrewise, so this is **not** a trivial $\mathbb{T}$-bundle). But it is not subtrivial; infinitely many irreducible representations of $\mathbb{T}$ appear.

3.3 Proposition Let G be compact. Any G-bundle over a homogeneous space is subtrivial.

Why is this true? Let H be a closed subspace of G. Let $\pi : H \to GL(V)$ any representation. Then $G \times_H V$ is a vector bundle over G/H. By a theorem of Mostow, there is a representation $\tilde{\pi} : G \to GL(\tilde{V})$ such that π is contained in the restriction of $\tilde{\pi}$ to H. Hence,

$$G \times_H V \subseteq G \times_H \tilde{V} \simeq G/H \times \tilde{V}. \qquad \square$$

3.4 Corollary Let G be compact. Any G-bundle on a compact space is subtrivial.

3.5 Clifford symbols Let V be a G-bundle over X that is Euclidean (*i.e.*, equipped with a real, G-invariant inner product). (*E.g.*, $G \subseteq \mathrm{Isom}(X)$, X a Riemannian manifold, $V = TX$.) A G-equivariant **Clifford symbol** for V is a pair (S, c) where S is a $\mathbb{Z}/2\mathbb{Z}$-graded, Hermitian G**-bundle** on X, and c is a bundle map

$$c : V \to \mathrm{End}_G(S)$$

such that

1. $c(\xi)$ is odd
2. $c(\xi)^* = c(\xi)$
3. $c(\xi)^2 = \|\xi\|^2$

for any $\xi \in V$.

Remark. 1. This gives, for each $x \in X$, a representation $c_x : Cl(V_x) \to \mathrm{End}(S_x)$.

2. Some authors (Roe, for example) require skew-adjointness $c(\xi)^* = -c(\xi)$.

3.6 Example Let X be Riemannian G-manifold. $V = TX$. $S = \bigwedge^*(TX \otimes_{\mathbb{R}} \mathbb{C})$. Let

$$c(\xi) := \varepsilon_\xi + \varepsilon_\xi^\dagger,$$

where ε_ξ denotes the map taking the wedge product with ξ.

3.7 Example $V = \mathbb{R}^2$. View this as a vector bundle over a point. Make it a $G = \mathbb{T}$-bundle by the representation

$$z = e^{i\theta} \mapsto R(2\theta), \text{rotation by } 2\theta.$$

Let $S = \mathbb{C} \oplus \mathbb{C}$ (first part even and second part odd grading) with the usual Hermitian structure. Let G act on S by the representation

$$z \mapsto \left(\begin{smallmatrix} z & 0 \\ 0 & \bar{z} \end{smallmatrix}\right) \in GL(2, \mathbb{C})$$

We define

$$c : \mathbb{R}^2 \to \mathrm{End}(\mathbb{C}^2), \qquad c(x, y) = \begin{pmatrix} 0 & x+iy \\ x-iy & 0 \end{pmatrix}.$$

Then c is **equivariant** in the sense that

$$c(g \cdot v) = g \circ c(v) \circ g^{-1}.$$

3.8 Triples Suppose X is a locally compact G-space. Consider the triple

$$(V^+, V^-, \phi)$$

where $V^\pm$ are subtrivial G-bundles over X, and ϕ is a G-bundle map $V^+ \to V^-$ which is an isomorphism off a compact set. (Or, we could require a single $\mathbb{Z}/2\mathbb{Z}$-graded bundle $V = V^+ \oplus V^-$ with an odd endomorphism ϕ.)

1. A triple is **degenerate** if ϕ is an isomorphism everywhere.
2. Two triples are **stably isomorphic** if they become isomorphic after adding a degenerate triple.
3. Two triples $(V_1^\pm, \phi_1)$, $(V_2^\pm, \phi_2)$ are **homotopic** if there is a triple over $X \times [0, 1]$ whose restrictions to $X \times \{0\}$ and $X \times \{1\}$ are stably isomorphic to $(V_1^\pm, \phi_1)$, $(V_2^\pm, \phi_2)$ respectively.

3.9 Definition (Segal) Denote by $L_G^0(X)$ the homotopy classes of triples. This is a group. Define

$$L_G^{-n}(X) \stackrel{\text{def}}{=} L_G^0(X \times \mathbb{R}^n).$$

3.10 Theorem

$$L_G^{-n}(X) \simeq K_G^{-n}(X).$$

Why is this true? Given $(V^{\pm}, \phi)$, first add a degenerate triple so that V^- becomes trivial. So V then extends to the one-point compactification X^+. Let U be a G-invariant open subset such that $\bar{U}$ is compact, outside of which ϕ is an isomorphism. Form the "clutching" bundle

$$V^+\restriction_U \cup_{\partial U} V^-\restriction_{X\setminus U}.$$

This is a trivial bundle outside of U so it extends to a G-bundle on X^+. Thus $[V^+\restriction_U \cup_{\partial U} V^-\restriction_{X\setminus U}]$ is an element of $K^0_G(X^+)$, and $[V^-]$ is also an element of $K^0_G(X^+)$. The difference $[V^+\restriction_U \cup_{\partial U} V^-\restriction_{X\setminus U}] - [V^-]$ belongs to $K^0_G(X)$. □

Exercise 7 Let $X = \mathbb{R}^2$, $V^{\pm} = X \times \mathbb{C}^2$. Show that $V^+\restriction_{\mathbb{D}} \cup_{\partial\mathbb{D}} V^-\restriction_{X\setminus\mathbb{D}}$ is isomorphic to the Hopf bundle H^* on $(\mathbb{R}^2)^+ \simeq S^2 \simeq \mathbb{C}P^1$.

Suppose we have a Clifford symbol (S, c) for a Euclidean G-bundle $\pi_V : V \to M$, where M is compact. Put $X = V$, and let $V^+ := \pi_V^*(S^+)$, $V^- := \pi_V^*(S^-)$, $\phi = c$. Since $c(\xi)^2 = \|\xi\|^2$, so $c(\xi)$ is invertible as long as $\xi \neq 0$, *i.e.*, c is invertible off the zero section of V. This gives a triple (V^+, V^-, ϕ).

$$\begin{array}{ccc} V & \longleftarrow & \pi^* S \\ {\scriptstyle \pi_V}\downarrow & & \downarrow \\ M & \longleftarrow & S \end{array}$$

3.11 Definition Let V be an even-dimensional Euclidean G-bundle over X. It is **G-K-orientable** if it admits a G-Clifford symbol (S, c) for V such that $\dim(S) = 2^{\dim(V)/2}$ (S is "irreducible").

3.12 Theorem If $\pi : V \to X$ is G-K-orientable, then the associated class to the triple obtained by the Clifford symbol yields a vector bundle $\pi^*S \to V$. This yields a class $\xi_V \in K_G^{\dim V}(V)$, called the **Thom class**. And the map

$$\begin{array}{ccc} K^*_G(X) & \to & K^{*+\dim V}_G(V) \\ a & \mapsto & \pi^*_V(a) \cdot \xi_V \end{array}$$

is an $R(G)$-module isomorphism.

4 Dirac Operators (Heath Emerson, June 17)

4.1 X, a complete Riemannian G-manifold.

- (S, c) a Clifford symbol for TX.

- ∇ a connection on S, that is
  (i) G-equivariant: $\nabla_{g\cdot X}(g \cdot s) = g \cdot \nabla_X s$.
  (ii) compatible with the Levi-Civita connection:
  $$\nabla_X(c(Y)s) = c(\nabla_X^{LC} Y)s + c(Y)\nabla_X s.$$
  (iii) compatible with the Hermitian structure on S
  $$\langle \nabla_X s_1, s_2 \rangle + \langle s_1, \nabla_X s_2 \rangle = X \langle s_1, s_2 \rangle .$$

Exercise 8 Prove that such a connection exists.

4.2 Definition Define a linear map D by the composition

$$\begin{array}{ccc} \Gamma_c(S) & \overset{D}{\dashrightarrow} & \Gamma_c(S) \\ \nabla \downarrow & & \uparrow \sqrt{-1}c \\ \Gamma_c(T^*X \otimes S) & \xrightarrow{\sim} & \Gamma_c(TX \otimes S) \end{array}$$

This is a G-equivariant operator on $\Gamma_c(S)$. Locally,

$$Ds(x) = \sum \sqrt{-1}c(e_i)\nabla_{e_i} s(x).$$

4.3 Example Let $S = \wedge^*(TX \otimes \mathbb{C})$ (since $\dim S = 2^{\dim_{\mathbb{R}} TX}$, this is not "irreducible," *i.e.*, K-orientable). Extend the Levi-Civita ∇ to $\wedge^* TX \otimes \mathbb{C}$. Then D in this case is

$$d + d^*$$

where d is the de Rham differential. This example is equivariant with respect to the group $G = \mathrm{Isom}(X)$.

4.4 Example The signature operator on an orientable smooth manifold X uses a grading that depends on the orientation. This gives a G-equivariant operator D which is equivariant with respect to $\mathrm{Isom}^+(X)$.

4.5 Example Let X be a complex manifold, admitting a complex structure. Let $S = \wedge^* TX$. Note that TX is already complex. $\dim_{\mathbb{C}} S = 2^{\dim_{\mathbb{C}} TX} = 2^{\dim_{\mathbb{R}} TX/2}$. Take $c(\xi) = \epsilon_\xi + \epsilon_\xi^*$. This Clifford symbol is irreducible. The operator D is the Dolbeault operator $\bar{\partial}$. D is equivariant with respect to any compact group of holomorphic maps of X.

For example, take $X = SL(n, \mathbb{C})/B$ where B is the subgroup of upper triangular matrices (the Borel subgroup). Then

$$X \simeq SU(n)/T$$

where T is the subgroup of diagonal matrices. Then $G = SU(n)$ acts by holomorphic maps on X.

Exercise 9 Take the above example of $SL(n, \mathbb{C})/B \simeq G/T$. Let $X = G/T$. The tangent bundle TX is a G-equivariant vector bundle on $X = G/T$. Express TX as an induced space from a representation of T. Do the same for S.

4.6 Example $X = S^1$, $S = S^1 \times \mathbb{C}$. Define

$$c : TS^1 = S^1 \times \mathbb{R} \to \mathrm{End}(S)$$

by

$$c(z, r) = \text{pointwise multiplication by } r.$$

(This is not a Clifford symbol in the usual sense because there is no grading on S; it is called an "odd Clifford symbol" because it leads to odd K-group class.) Then $D = \sqrt{-1}\frac{d}{d\theta}$, which acts on $C^\infty(S^1)$.

4.7 Theorem Let X be a complete Riemannian G-manifold, and (S, c) a G-Clifford symbol for TX, and let D be defined as in [4.2]. Then D has a unique self-adjoint extension $\bar{D}$ to $L^2(S)$. In particular, functional calculus gives a $*$-homomorphism

$$\begin{aligned} C_b(\mathbb{R}) &\to \mathcal{B}(L^2(S)), \\ f &\mapsto f(\bar{D}). \end{aligned}$$

Moreover, if X is compact, then $Sp(\bar{D}) \subseteq \mathbb{R}$ is discrete, all eigenvalues of $\bar{D}$ have finite-dimensional eigenspaces that consist of smooth sections of S.

Remark. The nth eigenvalue of $|D|$ is $\sim n^{1/\dim X}$. In particular, $f(\bar{D})$ is compact if $f \in C_0(\mathbb{R})$. And $\bar{D}$ is Fredholm, that is, $\ker(\bar{D}_+)$ and $\ker(\bar{D}_-)$ are finite-dimensional. They are also G-invariant subspaces of $L^2(S)$. ($D_+ := D\restriction_{L^2(S^+)} \dots$.) We define

$$\mathrm{ind}_G(D) \stackrel{\text{def}}{=} [\ker D_+] - [\ker D_-] \in R(G).$$

4.8 Let $V \in \mathsf{Vect}_G(X)$. Let (S, c) be a G-Clifford symbol for TX. Then $(S \otimes V, c \otimes \mathrm{id})$ is a new, "twisted" Clifford symbol for TX. The D operator in this case will be denoted by D_V.

4.9 Pairing with vector bundles So there is a pairing

$$\{\text{G-bundles on } X\} \times \{\text{ G-equivariant elliptic operators}\} \to R(G)$$

Atiyah's idea is that these should define "dual" theories. This pairing is analytically defined. But using symbols of operators, one can also understand this topologically. The connection between the topological and analytical descriptions of the pairing is Atiyah–Singer Index Theorem.

4.10 An application Let $G = SU(n)$. Let T be the maximal torus of diagonal matrices. Let $X = G/T$. Let $\bar{\partial}$ be the Dolbeault operator on X. It is G-equivariant.

The restriction map gives a map

$$R(G) \to R(T).$$

In fact, this map provides an isomorphism

$$R(G) \simeq R(T)^W.$$

onto the Weyl group invariant subspace.

Why is this true? Produce a map $R(T) \to R(G)$ such that $R(G) \to R(T) \to R(G)$ is the identity. We use **holomorphic induction**. Let $\alpha \in R(T)$. This, by induction, gives $V_\alpha \in \mathsf{Vect}_G(G/T)$. The index of the Dolbeault operator twisted by α gives an element of $R(G)$. It can be checked that this map restricted to W-invariants is inverse to the restriction map. □

5 Bivariant K-theory (Ralf Meyer, June 18)

5.1 I will introduce bivariant K-theory as a black box first. And try to convince you why it can be useful.

The idea is that bivariant K-theory is the home for maps between K-theory groups. We expect a natural map

$$KK(A, B) \to \mathrm{Hom}(K_*A, K_*B).$$

And we expect a composition product

$$KK(A, B) \otimes KK(B, C) \to KK(A, C).$$

What data leads to maps between K-theory groups?

- boundary maps of extensions
- index theory starts with maps $K^*(X) \to \mathbb{Z}$ (or $K^*_G(X) \to R(G)$) from ellipitic differential operators
- $*$-homomorphisms
- Bott periodicity, Thom isomorphisms
- twisting by a vector bundle (ring structure on $K^*(X)$).

All these constructions lift naturally to elements in KK.

5.2 But why not stick to $\mathrm{Hom}(K_*A, K_*B)$? Consider the following example. Let D be a Dirac type operator on a compact manifold X. Let Y be another compact manifold. $X \times Y \to Y$ is a bundle. And we would expect D to give a map $K^*(X \times Y) \to K^*(Y)$. And there is a map $K^*(X) \otimes K^*(Y) \to K^*(X \times Y)$. Since $K^*(X \times Y)$ also contains an additional piece, $\mathrm{Tor}(K^*X, K^*Y)$, a map $K^*(X) \to \mathbb{Z}$ does not directly induce a map $K^*(X \times Y) \to K^*(Y)$. For KK, however, there is an exterior product

$$KK(A, B) \to KK(A \otimes C, B \otimes C).$$

Once we view D as an element of $KK(C(X), \mathbb{C})$, we get an induced map $K^*(X, Y) \to K^*(Y)$.

5.3 Things get really interesting with equivariant KK, in particular for noncompact groups. Let G be a topological group. Let A, B be C^*-algebras with G-action. Then there is a $\mathbb{Z}/2\mathbb{Z}$-graded group $KK^G_*(A, B)$ which comes with a natrual map $KK^G_*(A, B) \to KK_*(A \rtimes G, B \rtimes G)$ (called the descent homomorphism).

Remark. The descent homomorphism is far from being isomorphic. Let G be compact. Let $A = B = \mathbb{C}$. Then $KK^G(\mathbb{C}, \mathbb{C}) = R(G)$.

$$\begin{array}{ccc} KK(C^*G, C^*G) & \xrightarrow{\simeq} & \mathrm{Hom}(R(G), R(G)) \\ {\scriptstyle \text{descent}} \uparrow & & \uparrow \\ KK^G(\mathbb{C}, \mathbb{C}) & \xrightarrow{\simeq} & \mathrm{Hom}_{R(G)}(R(G), R(G)) \end{array}$$

5.4 The use of KK-theory can be found in the structure of a proof of the Connes-Thom isomorphism and the Pimsner-Voiculescu exact sequence.

Recall Bott periodicity: $K^*(\mathbb{R}) \simeq K^{*+1}(\mathrm{pt})$. This may be proved by exhibiting $D \in KK_1(C_0\mathbb{R}, \mathbb{C})$, $\eta \in KK_1(\mathbb{C}, C_0\mathbb{R})$ and checking that they are inverse to each other.

Now, let $\mathbb{R}$ act on itself by translations, trivially on $\mathbb{C}$. It turns out, then, D and η are equivariant, *i.e.*, $D \in KK_1^{\mathbb{R}}(C_0\mathbb{R}, \mathbb{C})$, $\eta \in KK_1^{\mathbb{R}}(\mathbb{C}, C_0\mathbb{R})$. And D and η remain inverse to each other in $KK^{\mathbb{R}}$.

Now use exterior products in $KK^{\mathbb{R}}$: For any $\mathbb{R}$-C^*-algebra A, there is an isomorphism in $KK_1^{\mathbb{R}}(C_0\mathbb{R} \otimes A, \mathbb{C} \otimes A)$. Next, apply descent homomorphism to get an isomorphism in $KK_1((C_0\mathbb{R} \otimes A) \rtimes \mathbb{R}, (\mathbb{C} \otimes A) \rtimes \mathbb{R})$. $(C_0\mathbb{R} \otimes A) \rtimes \mathbb{R}$ is isomorphic to $\mathfrak{K}(L^2\mathbb{R}) \otimes A$ because the translation action is free and proper. This leads to the Connes-Thom isomorphism:

$$K_*(A \rtimes \mathbb{R}) \simeq K_{*+1}(A).$$

Remark.

1. There is an invertible element in $KK_0(C_0\mathbb{R}^2, \mathbb{C})$.
2. The class D should be a K-homology class on $\mathbb{R}$. It comes from the Dirac operator on $\mathbb{R}$, which is just $i\frac{d}{dx}$. Notice that D is translation invariant.
3. The class η comes from the C^*-algebra extension

$$C_0(\mathbb{R}) \rightarrowtail C_0([-\infty, \infty]) \twoheadrightarrow \mathbb{C}.$$

4. A similar technique applies to fundamental groups of negatively curved manifolds. Let M be a compact Riemannian manifold of negative curvature. Then its universal cover $\tilde{M}$ is diffeomorphic to $\mathbb{R}^n$, leading to an invertible element in $KK_n(C_0\tilde{M}, \mathbb{C})$. Analysis shows that this is $\pi_1(M)$-equivariant, with $\pi_1(M)$ acting by deck transformations on $\tilde{M}$, and that it remains invertible in $KK_n^{\pi_1(\tilde{M})}(C_0\tilde{M}, \mathbb{C})$.)

Exercise 10 Deduce the Pimsner-Voiculescu exact sequence, using an invertible element in $KK_1((A \otimes C_0\mathbb{R}) \rtimes \mathbb{Z}, A \rtimes \mathbb{Z})$. ($(A \otimes C_0\mathbb{R}) \rtimes \mathbb{Z}$ is Morita equivalent to the so-called mapping torus.)

5.5 Proposition If G is discrete, $KK^G(A, \mathbb{C}) \simeq KK(A \rtimes G, \mathbb{C})$.

KK_1 using C^*-algebra Extensions

5.6 Definition An extension $B \rightarrowtail E \twoheadrightarrow A$ is called **trivial** if it splits by a $*$-homomorphism $A \to E$.

Given two extensions of A by $B \otimes \mathfrak{K}$, say

$$B \otimes \mathfrak{K}(H_j) \rightarrowtail E_j \twoheadrightarrow A, \quad (j = 1, 2)$$

then there is a well-defined direct sum, call it E:

$$B \otimes \mathfrak{K}(H_1 \oplus H_2) \rightarrowtail E \twoheadrightarrow A.$$

$$E = \left\{ \begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix} \in E_1 \oplus E_2 \;\middle|\; e_1, e_2 \text{ have same image in } A \right\}.$$

An extension E is said to be **invertible** if there is another extension $E^\perp$ such that $E \oplus E^\perp$ is trivial.

$KK_1(A, B)$ is isomorphic to the group of homotopy classes of invertible extensions of A by $B \otimes \mathfrak{K}$, where homotopy is defined using invertible extensions of A by $C([0,1], B) \otimes \mathfrak{K}$. We may replace $B \otimes \mathfrak{K}$ above by $\mathfrak{K}(H_B)$ where H_B is a Hilbert module over B.

Note. Let $E \oplus E^\perp$ be trivial. Then we get a $*$-homomorphism

$$\rho : A \to (E \oplus E^\perp) \to \underbrace{\mathcal{M}(B \otimes \mathfrak{K})}_{\text{multiplier algebra}}$$

and a projection

$$p \in \mathcal{M}(B \otimes \mathfrak{K}), \quad p = \text{“}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\text{”}.$$

For $E := p\rho(A)p + p(B \otimes \mathfrak{K})p$ to be a C^*-algebra, we nee $[p, \rho(a)] \in B \otimes \mathfrak{K}$ for all $a \in A$. It turns out that invertibe extensions correspond to pairs $\rho : A \to \mathcal{M}(B \otimes \mathfrak{K})$ $*$-homomorphism, and $p \in \mathcal{M}(B \otimes \mathfrak{K})$ projection, such that $[p, \rho(a)] \in B \otimes \mathfrak{K}$ for all $a \in A$.

Now, replace p by $2p - 1 =: F$. Then $F^2 = 1$ and $F = F^*$. This is Kasparov's definition of KK_1. He also allows

$$(F - F^*)\rho(a) \in B \otimes \mathfrak{K},$$
$$(F^2 - 1)\rho(a) \in B \otimes \mathfrak{K}.$$

Remark. It follows from Stinespring's Theorem that an extension is invertible if and only if it has a completely positive contractive section.

6 (Ralf Meyer, June 21)

Hilbert modules are separable throughout this lecture.

6.1 Starting from invertible extensions, we get the following definition of $KK_1(A, B)$: Cycles consist of

1. a Hilbert B-module H (this can be replaced by $\mathcal{M}(B \otimes \mathfrak{K})$)

2. a $*$-homomorphism $\phi : A \to \mathfrak{B}(H)$
3. an operator $F \in \mathfrak{B}(H)$

such that

$$[F, \phi(a)], \quad (1 - F^2)\phi(a), \quad (F - F^*)\phi(a)$$

are compact (are in $B \otimes \mathfrak{K}$ if we replace H by $\mathcal{M}(B \otimes \mathfrak{K})$). $KK_1(A, B)$ is the group of homotopy classes of cycles.

6.2 Example 1. A homotopy is a cycle for $KK_1(A, C([0, 1], B))$.

2. If $B = \mathbb{C}$, $KK_1(A, \mathbb{C})$ is called the K-homology of A. Here, H is a Hilbert space.

Exercise 11 Check that $KK_1(\mathbb{C}, \mathbb{C}) = 0$.

Hint: $A = B = \mathbb{C}$. There is a homotopy between (H, ϕ, F) and $(\phi(1)H, 1, \phi(1)F\phi(1))$. (Use $\{ f : [0, 1] \to H \mid f(0) \in \phi(1)H \}$ as $C([0, 1])$-Hilbert module.)

Suppose $\phi(1) = 1$. Then the conditions on F say that $F^2 - 1$ and $F - F^*$ are compact. We may replace F by $\frac{1}{2}(F + F^*)$, which gives $F = F^*$.

6.3 Definition A cycle with $F^2 - 1 = 0$, $F = F^*$, $[F, \phi(a)] = 0$ is said to be **degenerate**. Degenerate cycles give the zero class in KK.

6.4 Theorem $KK_1(\mathbb{C}, B) \simeq K_1(B)$.

Why is this true? First, achieve $\phi(1) = 1$ as above. Cycles are Hilbert B-modules H with $F \in \mathfrak{B}(H)$ such that $F = F^*$, $F^* - 1$ are compact.

By adding on a degenerate cycle we may achieve $H = B \otimes \ell^2\mathbb{N}$ (this owes to [6.5]). Also, adding compact operators to F does not change the homotopy class (use the homotopy $F + tS, t \in [0, 1]$).

Therefore,

$$KK_1(\mathbb{C}, B) = \pi_0(\{ \dot{F} \in \mathfrak{B}(B \otimes \mathfrak{K})/(B \otimes \mathfrak{K}) \mid \dot{F}^2 = 1, \dot{F} = \dot{F}^* \}).$$

Now, $\frac{1}{2}(\dot{F} + 1) = \dot{p}$ is a projection. This leads to

$$KK_1(\mathbb{C}, B) = K_0(\mathcal{M}(B \otimes \mathfrak{K})/(B \otimes \mathfrak{K})).$$

The K-theory long exact sequence and $K_*(\mathcal{M}(B \otimes \mathfrak{K})) = 0$ finish the proof. □

6.5 Theorem (Kasparov Stabilization Theorem) For any separable Hilbert B-module H, $H \oplus (B \otimes \ell^2\mathbb{N}) \simeq B \otimes \ell^2\mathbb{N}$.

6.6 To get $KK_0(A, B)$, we must add some grading information: Let H be $\mathbb{Z}/2\mathbb{Z}$-graded, let $\phi : A \to \mathfrak{B}(H)$ be grading-preserving, and let $F \in \mathfrak{B}(H)$ be odd. Using these cycles and corresponding homotopies, we get $KK_0(A, B)$.

Remark. $KK_0(\mathbb{C}, B) \simeq K_0(B)$. In particular, $KK_0(\mathbb{C}, \mathbb{C}) \simeq \mathbb{Z}$. The isomorphism maps a cycle $(H_+ \oplus H_-, F)$ with $\varphi = 1$, $F = \begin{pmatrix} 0 & b \\ b^* & 0 \end{pmatrix}$ to the index of the Fredholm operator b.

6.7 How does an elliptic operator give a KK-class? Let X be a compact manifold, and let $A = C(X)$. Let

$$D : \Gamma(E^+) \to \Gamma(E^-)$$

be an elliptic differential operator of order 1, where $E^\pm$ are vector bundles over X. We construct a cycle for $KK_0(C(X), \mathbb{C})$. Let

$$H = L^2(X, E^+) \oplus L^2(X, E^-).$$

This is $\mathbb{Z}/2\mathbb{Z}$-graded. Let

$$\phi : C(X) \to \mathfrak{B}(H).$$

by pointwise multiplication operators. Finally,

$$F = \chi(D)$$

where $\chi : \mathbb{R} \to [-1, 1]$ satisfies $\lim_{x\to\pm\infty} \chi(x) = \pm 1$ and $\chi(-x) = -\chi(x)$, say, take the arctangent.) Here, it is crucial that D is (essentially) self-adjoint.

Then, $F^2 - 1 = (\chi^2 - 1)(D)$ is compact because $\chi^2 - 1$ is in $C_0(\mathbb{R})$ and D has compact resolvent. Next, $F = F^*$. And D is of order 1, so $[D, \phi(a)]$ is bounded for a smooth function a. It follows that $[F, \phi(a)]$ is compact.

6.8 In the equivariant case, for a locally compact group G acting continuously on A and B, we define $KK_*^G(A, B)$ by requiring an (even) continuous G-action on H, a G-equivariant ϕ, and $(g \cdot F - F)\phi(a)$ to be compact for $a \in A$. (Note that $g \mapsto g \cdot F - F$ is continuous.)

6.9 Kasparov product The Kasparov product is a rather deep construction. Given cycles for $KK_i^G(A, B)$ and $Kk_j^G(B, C)$, Kasparov describes a cycle for $KK_{i+j}^G(A, C)$ and shows that the product so defined is associative and has other nice properties. The main difficulty is to construct the operator F in a Kasparov product. Later, Connes and Skandalis described F by writing down conditions it should satisfy, making the proofs of properties of KK more transparent. (Reference: Blackadar, K-*theory for Operator Algebras*)

6.10 Example Easy cases of the Kasparov product. Let $\phi : A \to B$ be a $*$-homomorphism. It yields a class for $KK_0(A, B)$ by

$$H_+ = B, \quad H_- = 0,$$
$$\phi : A \to B \trianglelefteq \mathfrak{B}(B),$$
$$F = 0.$$

The Kasparov product with such classes reduces to the obvious functoriality of $KK_*(A, B)$ for $*$-homomorphism. Let (H, ϕ, F) be a cycle for $KK_*(A, B)$, $\psi : A' \to A$, then

$$\psi^*(H, \phi, F) = (H, \phi \circ \psi, F)$$

is a cycle of $KK_*(A', B)$. Given $\xi : B \to B'$) define

$$\xi_*(H, \phi, F) = (H \otimes_\xi B', \phi \otimes \mathrm{id}, F \otimes 1).$$

6.11 Let p be a projection in A. View p as a $*$-homomorphism $\mathbb{C} \to A, 1 \mapsto p$. Given a cycle (H, ϕ, F) for $KK_*(A, B)$, the product of it with $[p] \in K_0(A) = KK_0(\mathbb{C}, A)$ is the cycle $(H, \phi \circ \pi, F)$ for $KK_*(\mathbb{C}, B) \simeq K_*(B)$.

7 (Ralf Meyer, June 22)

7.1 Theorem $KK_*^G(\mathbb{C}, A) \simeq K_*(G \ltimes A)$ for compact G.

Why is this true? If G is compact, then we may assume the operator F in a cycle for $KK^G(\mathbb{C}, A)$ to be exactly G-equivariant because $\int_G gF\, dg$ is a compact perturbation of F. The Hilbert module may be taken to be $L^2G \otimes \ell^2\mathbb{N} \otimes A$.

The theorem follows from the following fact:

$$\mathfrak{K}(L^2G \otimes \ell^2\mathbb{N} \otimes A)^G = G \ltimes A \otimes \mathfrak{K}(\ell^2\mathbb{N}).$$

For example, $C^*G \simeq \mathfrak{K}(L^2G)^G$. □

Remark. If G is noncompact we may interpret

$$K_*(G \ltimes A) \simeq KK_*^G(X, C_0(X) \otimes A)$$

if there is a G-map $\mathrm{Prim}(A) \to X$ for a proper cocompact Hausdorff G space X.

7 (RALF MEYER, JUNE 22)

Duality in KK

7.2 Motivation The K-homology of a compact manifold X should be isomorphic to the K-theory of TX. Even more, the following holds: for C^*-algebras A and B,

$$\mathrm{KK}(A \otimes C(X), B) \simeq \mathrm{KK}(A, C_0(TX) \otimes B).$$

In particular,

$$\mathrm{KK}(C(X), \mathbb{C}) \simeq \mathrm{KK}(C, C_0(TX)).$$

How can one prove this statement? If $A = \mathbb{C}$, $B = C(X)$, then the statement becomes

$$\mathrm{KK}(C(X), C(X)) \simeq \mathrm{KK}(\mathbb{C}, C_0(TX) \otimes C(X)).$$

So there must be some element β on the right that corresponds to 1 on the left. If we take $A = C_0(TX)$ and $B = \mathbb{C}$, then

$$\mathrm{KK}(C(X) \otimes C_0(TX), \mathbb{C}) \simeq \mathrm{KK}(C_0(TX), C_0(TX)).$$

There must be some element α on the left that corresponds to 1 on the right.

7.3 Theorem Let $\alpha \in \mathrm{KK}^G(D^* \otimes D, \mathbb{C})$ and $\beta \in \mathrm{KK}^G(\mathbb{C}, D \otimes D^*)$. Then the following are equivalent:

(i) The map

$$\mathrm{KK}^G(A \otimes D, B) \to \mathrm{KK}^G(A, B \otimes D^*)$$

obtained by

$$\mathrm{KK}^G(A \otimes D, B) \xrightarrow{-\otimes D^*} \mathrm{KK}^G(A \otimes D \otimes D^*, B \otimes D^*) \xrightarrow{(\mathrm{id}_A \otimes \beta)^*} \mathrm{KK}^G(A, B \otimes D^*)$$

is an isomorphism.

(ii) The following compositions are identities:

$$D \xrightarrow{\mathrm{id}\otimes\beta} D \otimes D^* \otimes D \xrightarrow{\alpha\otimes\mathrm{id}} D,$$

$$D^* \xrightarrow{\beta\otimes\mathrm{id}} D^* \otimes D \otimes D^* \xrightarrow{\mathrm{id}\otimes\alpha} D^*.$$

(One must read the above compositions as elements of $\mathrm{KK}^G(D, D)$ and $\mathrm{KK}^G(D^*, D^*)$.)

7.4 How can we produce α and β for $C(X)$ and $C_0(TX)$? More generally, how can we generate interesting KK-classes between commutative C^*-algebras?

7.5 Geometric sources of elements in $KK^G(C_0(X), C_0(X))$

(i) a proper G-map $f : Y \to X$ induces $f^* : C_0(X) \to C_0(Y)$.

(ii) a G-equivariant vector bundle E over X gives a class in $KK^G(C_0(X), C_0(X))$; the space $\Gamma_0(E)$ of C_0-sections of E with Hilbert $C_0(X)$-module structure from a fibrewise inner product; $C_0(X) \to \mathfrak{K}(\Gamma_0 E)$ by multiplication operators, F irrelevant, say $F = 0$; grading is trivial.

(iii) if X is open in Y, $C_0(X) \to C_0(Y)$ as an ideal.

(iv) if X is the total space of a K-oriented vector bundle over Y (or vice versa), then the Thom isomorphism gives an invertible element in $KK^G(C_0X, C_0Y)$.

It turns out that these are almost all the ingredients we need. The main modification we need: If $f : Y \to X$ is not proper, we may still combine it with a K-theory class on Y with X-compact support to get a class in $KK^G(C_0X, C_0Y)$.

First small improvement: We may replace vector bundles by K-theory classes. $\xi \in K^G_*(X)$ gives class in $KK^G_*(C_0(X), C_0(X))$.

Consider a triple $(E_+, E_-, \phi : E_+ \to E_-)$ where $E_\pm$ are G-vector bundles over Y, ϕ a G-equivariant vector bundle map. Let $H_\pm = \Gamma_0(E_\pm)$, a $\mathbb{Z}/2\mathbb{Z}$-graded Hilbert $C_0(Y)$-module, with $C_0(X)$ acting by pointwise multiplication via $f : Y \to X$. Let $F : H_+ \to H_-$ be the pointwise application of ϕ, and let $F^* : H_- \to H_+$ be the adjoint of that. We need $(1 - F^2) \circ M_{h \circ f}$ compact for all $h \in C_0(X)$. This holds if ϕ is unitary outside an X-compact subset. In our case, $\mathfrak{B}(\Gamma_0(E)) = C_b(Y, \mathrm{End}(E))$ and $\mathfrak{K}(C_0(E)) = C_0(Y, \mathrm{End}(E))$.

Let $K^*_{G,X}(Y)$ be the K-theory of Y with X-compact support, defined by triples (E_+, E_-, ϕ) with ϕ unitary outside an X-compact support. This gives classes in $KK^G(C_0X, C_0Y)$ by the previous construction.

The Topological Index Map of Atiyah–Singer

7.6 Let X be a compact manifold. Embed X into $\mathbb{R}^n$. This induces an embedding $TX \hookrightarrow T\mathbb{R}^n = \mathbb{C}^n$. Any embedding has a tubular neighbourhood (*i.e.*, the normal bundle). Some open neighbourhood of TX in $\mathbb{C}^n$ is diffeomorphic to the total space of a vector bundle N over X (namely, the normal bundle). N is canonically K-oriented (there is an appropriate Clifford symbol). So then,

$$K^*(TX) \xrightarrow{\text{Thom iso}} K^*(N) \hookrightarrow K^*(\mathbb{C}^n) \xrightarrow{\text{Bott periodicity}} K^*(pt) = \mathbb{Z}.$$

This series of compositions is the topological index map of Atiyah and Singer.

8 Baum-Connes Conjecture (Heath Emerson, June 23)

8.1 CV of NCG Genuine contribution of NCG to other fields.

- Novikov conjecture, coarse geometry. (Positive scalar curvature problem.)
- Hyperbolic dynamical systems, hyperbolic groups, hyperbolic foliations.

The above two are actually connected.

- Representation theory: Connes-Kasparov conjecture.
- Pure topology
  - Equivariant Euler characteristics (for proper actions of discrete groups).
  - Twisted K-theory
  - Orbifolds

8.2 Baum-Connes I Let G be a locally compact group. Let X be a proper G-space. (*I.e.*, $\{ g \in G \mid g(K) \cap L \neq \varnothing \}$ is compact for all K, L compact in X. "X is big enough") Let A, B be C_0-sections of a continuous field of C^*-algebras over X with G-action. (*E.g.*, $C_0(X)$, $C_0(Cl(TX))$, $C_0(X, D)$,... a crossed product $G \ltimes A$ for such A is again of Type I if A is.)

The groupoid equivariant KK-theory $KK^{G \ltimes X}(A, B)$ has "continuous" families $\{ (H_x, F_x) \mid x \in X \}$ as cycles, where (H_x, F_x) is a cycle for $KK(A_x, B_x)$ for all x. The equivariance condition is that, for any $g \in G$, $g : H_x \to H_{g \cdot x}$ and we must have $gF_x g^{-1} - F_{g \cdot x}$ compact for all $x \in X$. (This is not much of an equivariance; consider for example the case when $X = G$.)

8.3 Example 1. Let $A = B = C_0(X)$.

$$KK_*^{G \ltimes X}(C_0(X), C_0(X)) =: RK_G^*(X)$$

is the representable G-equivariant K-theory of X, which is isomorphic to the space of homotopy classes of G-maps from X to the space of Fredholm operators on $L^2(G) \otimes \ell^2(\mathbb{N})$.

2. If X, G are compact, then $RK_G^*(X) \simeq K_G^*(X)$.

8.4 Contravariant Baum-Connes For any G-C^*-algebras A and B, consider the map

$$P_{EG}^* : KK^G(A, B) \to KK^{G \ltimes EG}(C_0(EG, A), C_0(EG, B))$$

where EG is the universal proper G-space

$$P_{EG}^*[(H, F)] = [\{ (H_x := H, F_x = F) \mid x \in EG \}].$$

"Baum-Connes" (not the original one): P^*_{EG} is an isomorphism. This is false, but the statement is true for many cases. For example, consider the following example:

8.5 Example $G = F_2$ and let T be the universal covering space, which is a tree. Let $H = \ell^2(T^{(\text{vertex})}) \oplus \ell^2(T^{(\text{edge})})$. This is $\mathbb{Z}/2\mathbb{Z}$-graded. Define $b(\delta_{\text{vertex } v}) = \delta_{\text{edge } s(v)}$, where $s(v)$ is the edge adjacent to v that leads to the base point, and $b(\delta_{\text{base point}}) = 0$.

1. Note that the Fredholm index of b is 1.
2. $gbg^{-1} - b$ is compact (even finite rank) for all $g \in F_2$. This is related to the negative curvature of trees.

Question: Is $\gamma = [(H, F)]$ equal to 1 in $KK^{F_2}(\mathbb{C}, \mathbb{C})$? (1, by the way, is equal to $[(H, F)] = [(\mathbb{C} \oplus 0, 0)]$.

Here, EF_2 can be taken to be $|T|$, that is, the total space of T. Check that $P^*_{EG}(\gamma) = 1$ in $KK^{F_2 \ltimes EF_2}(C_0(EF_2), C_0(EF_2)$.

Note. The message is that P^*_{EG} for a torsion-free group G is mapping equivariant problem to nonequivariant problems.

8.6 Theorem The map P^*_{EG} is invertible for $G = F_2$.

Why is this true? This follows from Julg-Valette (*i.e.*, $\gamma = 1$ in $KK^G(\mathbb{C}, \mathbb{C})$) or from Higson-Kasparov. □

8.7 Corollary Let A be a F_2-C^*-algebra that satisfies the UCT. Then so does $A \rtimes_r F_2$.

Remark. This is also true for $A \rtimes F_2$.

Why is this true? γ factors as $\alpha \otimes_p \beta$ where $\alpha \in KK^G(\mathbb{C}, P)$ and $\beta \in KK^G(P, \mathbb{C})$, and p is a proper G-$C^*$-algebra. Then $P \rtimes G$ satisfies UCT. $P \otimes A \rtimes G$ also satisfies UCT for all A. Then, $\alpha \otimes 1_A$ is invertible in $KK^G(A, A \otimes P)$, and so is its image under the descent to $KK(A \rtimes G, A \otimes P \rtimes G)$. But $A \rtimes G$ and $A \otimes P \rtimes G$ are KK-equivaraint. □

8.8 Example Let Γ be a uniform lattice in $SL(2, \mathbb{R})$. Then $E\Gamma$ is equal to $\mathbb{H}^2$. Higson-Kasparov says that $\gamma = 1$ in this case too. Now, Γ acts on $\partial\mathbb{H}^2 = \mathbb{R} \cup \{\infty\}$. It preserves the Riemannian metric. $C(\partial\mathbb{H}^2) \rtimes \Gamma$ is purely infinite and nuclear.

It is hard to construct a Dirac operator, but, there does exist a Γ-equivariant *family* $\{ \langle \, , \rangle_x \mid x \in \mathbb{H}^2 \}$ of Riemannian metrics on $\partial\mathbb{H}^2$ because $\mathbb{H}^2 \times \mathbb{H}^2 \simeq S\mathbb{H}^2$, the unit tangent bundle to $\mathbb{H}^2$, and Γ acts isometrically on $\mathbb{H}^2$ and $S\mathbb{H}^2$. So we can build a Γ-equivariant Dirac class in $KK^{\Gamma \ltimes \mathbb{H}^2}(C_0(\mathbb{H}^2 \times \partial\mathbb{H}^2), C_0(\mathbb{H}^2))$, which is isomorphic to (by Baum-Connes) $KK^\Gamma(C(\partial\mathbb{H}^2), \mathbb{C}) \simeq K^0(C(\partial\mathbb{H}^2) \rtimes \Gamma)$.

Remark. The map P^*_{EG} is not invertible for cocompact lattices in $SL(2, \mathbb{R})$.

8.9 Theorem (Higson-Kasparov-Tu) If G (discrete) acts amenably on a commutative C^*-algebra A, then

$$P^*_{EG} : KK^G(A, B) \to KK^{G \ltimes EG}(C_0(EG, A), C_0(EG, B))$$

is an isomorphism for any B.

8.10 Corollary $A \rtimes_r G$ satisfies UCT.

8.11 Remark If G uniformly embeds in a Hilbert space, then P^*_{EG} is surjective. This implies the Novikov conjecture.

8.12 Baum-Connes II Let G be a locally compact group. Let X be a smooth G-manifold. Let M be a proper smooth G-mailfold such that $G \backslash M$ is compact. Let $\pi : M \to X$ be a G-equivariant K-oriented submersion. Then we can build a family $\{D_x\}_{x \in X}$ of Dirac operators along the fibres of π. This gives a class $\pi!$ in $KK^G(C_0(M), C(X))$. Under the higher index map, this is mapped into $\mathrm{ind}_G(\pi!)$ in $K_*(C(X) \rtimes G)$.

Baum-Connes: All K-theory classes for $C(X) \rtimes G$ arise in this way, if we also twist by G-vector bundles on M.

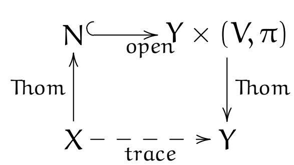
Exercise 12 Show that $[p_G]$ in $K_0(C(S^1) \rtimes_{R_\theta} \mathbb{Z})$ arises in this say.

9 Wrong-Way Maps in KK-theory (Ralf Meyer, June 24)

9.1 Definition Let X and Y be G-spaces, where G is compact. A **normally nonsingular map** from X to Y consists of

1. a subtrivial G-bundle N over X,
2. a finite-dimensional representation $\pi : G \to GL(V)$ of G,
3. an open embedding $f : N \to Y \times V$.

Such a map is said to be K-oriented if N and (V, π) are K-oriented.



9.2 A K-oriented normally nonsingular map $f : X \to Y$ induces a wrong-way map

$$f_! : K^*_G(X) \to K^{*+d}_G(Y), \quad d := \dim N - \dim V,$$

by composing the Thom isomorphisms for $N \to X$ and $Y \times V \to Y$ and the wrong-way map for the open embedding $N \hookrightarrow Y \times V$.

An important example is the Atiyah–Singer topological index map

$$\begin{array}{ccc} N & \hookrightarrow & \mathbb{C}^n \\ \downarrow & & \downarrow \\ TX & & pt \end{array}$$

9.3 Let X, Y be smooth G-manifolds. Then we may require the vector bundle N to have a smooth structure and the open embedding to be a diffeomorphism onto its range. We want to lift a smooth map $f : X \to Y$ to a smooth normally nonsingular map.

9.4 Theorem (Mostow) A G-equivariant embedding of X into a finite-dimensional representation of G exists if and only if the G-action on X has finite orbit type (*i.e.*, only finitely many conjugacy classes of subgroups of G occur as stabilisers of points in X).

Remark. If X is compact, then X has finite orbit type.

As a counterexample, the $\mathbb{T}$-space

$$\bigsqcup_{n=1}^{\infty} \mathbb{T}/\{\, e^{2\pi i k/n} \mid k = 0, \dots, n-1 \,\}$$

does not embed $\mathbb{T}$-equivariantly into any $\mathbb{T}$-representation.

If we replace compact groups by groupoids (or noncompact groups), then analogues of this embedding result may fail.

9.5 Let $h : X \hookrightarrow V$ be a smooth G-equivariant embedding into a representation of G. Then $(f, h) : X \to Y \times V$ is also an embedding. The tubular neighbourhood theorem provides an open embedding $N \hookrightarrow V \times Y$. This yields a normally nonsingular map with trace f. This construction is unique for a suitable notions of equivalence which involves lifting by a representation (W, π) of G:

$$(N \oplus X \times W) \xrightarrow{f \times \mathrm{id}_W} (V \oplus W) \times Y$$

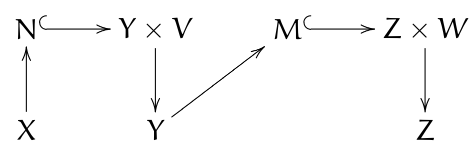
$$X \qquad\qquad\qquad Y$$

and an isotopy: a normally nonsingular map over $[0,1]$ from $X \times [0,1] \to Y \times [0,1]$.

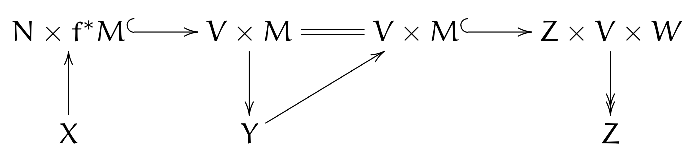
9.6 Theorem If there is an embedding $X \hookrightarrow (V, \pi)$ (so X has finite orbit type), then equivalence classes of normally nonsingular maps correspond to homotopy classes of smooth maps.

9.7 It seems, then, that there is almost no difference between smooth maps and normally nonsingular maps. But if we worked with smooth maps instead, then all our results would require technical assumptions about equivariant embeddings. And we would have to construct wrong-way maps directly for smooth maps, which would still require a factorisation as above. We found it more convenient to introduce normally nonsingular maps to avoid technical problems. First we only defined smooth normally nonsingular maps, but then it is immediately observed that smoothness is no longer helpful once the factorisation above is part of the definition. Hence we dropped smoothness. However, the existence of a normally nonsingular map from X to the point implies some smooth structure on X: there must be a smooth structure on $X \times \mathbb{R}^N$ for some N.

9.8 Composition of Normally Nonsingular Maps



We can pullback M over $Y \times V$, over N, and all the way over X. Lift the second map along V, and lift the first map along the pullback of M over X. So we get



The resulting normally nonsingular map from X to Z is well-defined up to homotopy, and this defines the composition of normally nonsingular maps. The composition induces the composition of the maps $K^*_G(X) \to K^*_G(Y) \to K^*_G(Z)$ on K-theory because lifting does not alter the induced map on K-theory and neither does going up and down in a K-oriented vector bundle by the Thom isomorphism.

Actually, the wrong-way map construction also yields classes in $KK^G_*(C_0X, C_0Y)$, and the map from normally nonsingular maps to bivariant K-theory is functorial for the composition and the Kasparov product.

The Atiyah–Singer Index Theorem for Families follows from the following:

9.9 Theorem Let $\pi : X \to Y$ be a G-equivariant K-oriented submersion. Then $\pi_! \in \mathrm{KK}^G_d(C_0X, C_0Y)$ where $d = \dim X - \dim Y$ is equal to the class of the family of Dirac operators along the fibres of π.

9.10 What remains to be done to prove this theorem? Lift the submersion π to a normally nonsingular map

$$\begin{array}{ccc} N & \overset{f}{\hookrightarrow} & Y \times V \\ \downarrow{\scriptstyle\sigma} & & \downarrow{\scriptstyle\tau} \\ X & \overset{\pi}{\longrightarrow} & Y \end{array}$$

and let $\sigma \colon N \to X$ be the vector bundle projection. We may arrange for $\tau \circ f = \pi \circ \sigma$. For vector bundle projections, the construction of the Thom isomorphism in bivariant K-theory already ensures $\sigma_! = [\not{D}_\sigma]$, and this class is invertible. The map $\tau \circ f = \pi \circ \sigma$ is a vector bundle projection followed by an open projection. It is not hard to show that the Dirac class is compatible with open embeddings. Hence we also get

$$(\pi \circ \gamma)_! = [\not{D}_{\pi\circ\sigma}].$$

Thus the index theorem follows from

$$[\not{D}_{\pi\circ\sigma}] = [\not{D}_\pi] \circ [\not{D}_\sigma].$$

This requires us to compute a certain Kasparov product explicitly.

10 KK-theory via correspondences (Ralf Meyer, June 25)

10.1 We want to describe $\mathrm{KK}^G(C_0X, C_0Y)$ by geometric cycles, where G is a compact Lie group, and X, Y are compact G-spaces. (X is a smooth G-manifold.) This idea goes back to Paul Baum (defined for K-homology) in the 1980s and Connes-Skandalis (generalized to bivariant K-theory).

Today's results are contained in [6,7].

$$\begin{array}{ccccc} & & (M, \xi) & & \\ & \overset{b}{\swarrow} & & \overset{f}{\searrow} & \\ X & & & & Y \end{array}$$

where $\xi \in K^*_{GX}(M)$, $b : M \to X$ is a G-map, $f : M \to Y$ is a G-equivariant normally nonsingular K-oriented map.

$$\begin{array}{ccc} \mathrm{KK}^G(C_0X, C_0M) \otimes \mathrm{KK}^G(C_0M, C_0Y) & \to & \mathrm{KK}^G_*(C_0X, C_0Y) \\ (b, \xi)^* \otimes f_! & \mapsto & (b, \xi)^* \circ f_! \end{array}$$

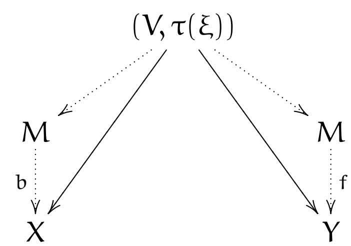
When do two such cycles (M, ξ, b, f) give the same element of $KK^G(C_0X, C_0Y)$? The equivalence relation is generated by three elementary moves:

1. Direct sum–disjoint union: for two cycles with the same M, f, b:

$$(M, \xi_1 + \xi_2, b, f) \sim (M \sqcup M, \xi_1 \sqcup \xi_2, b \sqcup b, f \sqcup f).$$

2. Bordism: First in a special case: let M, Y be smooth manifolds, f smooth. A bordism is like a cycle, but M is replaced by a manifold with boundary W and f by a smooth map $W \to Y$. Here, $W = \partial_0 W \sqcup \partial_1 W$. (Think of the pants diagram; the waist is $\partial_0 W$, and the foot part is $\partial_1 W$.) Restrictions of f, b, ξ to $\partial_0 W$ and $\partial_1 W$ give cycles. They are said to be bordant. This can be carried over to the case of normally nonsingular maps as well.

3. Thom modification: Let (M, ξ, b, f) be a cycle. Let V be a G-equivariant K-oriented vector bundle over M. Then (M, ξ, b, f) is equivalent to



where $\tau : K^*_G(M) \to K^*_G(V)$, the Thom isomorphism. (Paul Baum compactifies V because he requires b to be a proper map. Since we allow non-proper b, we may use the total space itself, which makes some constructions considerably easier.)

Since the wrong way map for $V \twoheadrightarrow M$ is the inverse Thom isomorphism, this cycle has the same image in KK^G.

10.2 Theorem Let $\widehat{KK}^*_G(X, Y)$ be the set of equivalence classes of cycles modulo the equivalence relation generated by the above elementary moves. If there is a normally nonsingular map from X to pt (or $[0, 1]$) then the map

$$\widehat{KK}^*_G(X, Y) \to KK_G(C_0X, C_0Y)$$

is an isomorphism.

Remark. This applies if X is a smooth G-manifold with boundary and finite orbit type.

Why is this true? First, look at the case $X = \mathrm{pt}$. Then $KK^*_G(C_0\mathrm{pt}, C_0Y) = K^*_G(Y)$.

Let (M, ξ, b, f) be a cycle for $X = \mathrm{pt}$ and some Y. We may ignore $b : M \to \mathrm{pt}$. Remember that f is a triple:

$$\begin{array}{ccc} N & \xrightarrow[\text{open}]{\tilde{f}} & Y \times V \\ \downarrow & & \downarrow \pi \\ M & \underset{\mathrm{trace}}{\dashrightarrow} & Y \end{array}$$

We do a Thom modification along N to get a cycle $(N, \hat{\xi}, \mathrm{const}, \pi \circ \tilde{f})$. This is bordant to a cycle $(Y \times V, \tilde{f}_!(\tilde{\xi}), \mathrm{const}, \pi)$. Now do an inverse Thom modification to get a cycle $(Y, \tilde{\tilde{\xi}}, \mathrm{const}, \mathrm{id})$, where $\tilde{\tilde{\xi}} \in K^*_G(Y)$. Thus every cycle is equivalent to one of this special form. Furthermore, two cycles of this special form are only equivalent if they have the same element in $K^*_G(Y)$.

Next, we use duality isomorphisms to reduce the bivariant case to the K-theory case just treated: For compact X,

$$\begin{array}{ccc} \widehat{KK}^G(X, Y) & \simeq & \widehat{KK}^G(\mathrm{pt}, Y \times TX) \\ \downarrow & & \downarrow \sim \\ \widehat{KK}^G(C_0X, C_0Y) & \simeq & KK^G(\mathbb{C}, C_0(TX) \otimes C_0(Y)). \end{array}$$

(If X is not compact, we need to consider $\widehat{KK}^{G \ltimes X}(X, Y \times TX)$, the G-equivariant K-theory of $Y \times TX$ with X-compact support, instead of $\widehat{KK}^G(\mathrm{pt}, Y \times TX)$.) So it only remains to prove the duality in $\widehat{KK}^G$ between X and TX. This implies duality in KK^G between C_0X and C_0Y because we may map the unit and counit of the adjointness in $\widehat{KK}^G(\mathrm{pt}, X \times TX)$ and $\widehat{KK}^G(X \times TX, \mathrm{pt})$ to classes in KK^G, and these still satisfy the relevant conditions because the map from $\widehat{KK}^G$ to KK^G is a functor and maps products to tensor products.

Let α be the class

$$\mathrm{pt} \longleftarrow (X, 1) \xrightarrow{f} X \times TX$$

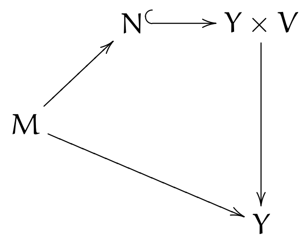
where the map f sends x to $(x, x, 0)$. Because X is compact, $X \to \mathrm{pt}$ is proper, and $1 \in K^*_{G,\mathrm{pt}}(X) = K^*_G(X)$. Since TX is almost complex, the smooth map f is K-oriented.

Let β be the class

$$X \times TX \xleftarrow{b} (TX, 1) \longrightarrow \mathrm{pt}$$

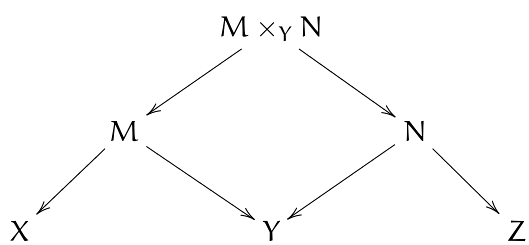
where b maps (x, ξ) to (x, x, ξ). This is a correspondence as well.

We still have to discuss the composition product of correspondences. This should be constructed as an intersection product. But this requires that the two maps $M \to Y \leftarrow N$ are "transverse." What to do in general? – Thom modification. The map $N \to Y$ is a very special "submersion," therefore it is transverse to anything. We can use Thom modification to replace M by N.



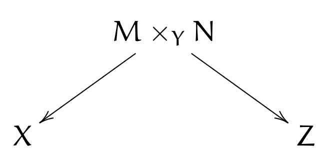
Thus it suffices to study the composition product for cycles where $f\colon M \to Y$ is an open embedding followed by a trivial vector bundle projection. We may also simplify the equivalence relation for such special correspondences, leading to a model for bivariant K-theory with rather simple geometric cycles and relations. Since it is not so easy to bring more general correspondences into this form – this requires embedding manifolds and constructing tubular neighbourhoods – we still need the definition above to actually construct correspondences. But for formal arguments, it may be easier to restrict attention to special cycles. □

10.3 Product in $\widehat{KK}^G$ The product ought to be an intersection product



where

$$M \times_Y N = \{ (m, n) \in M \times N \mid \text{same image in } Y \}.$$



with an appropriate K-theory class on $M \times_Y N$ should be the product. But for this to work, we need the coordinate projection $M \times_Y N \to N$ to be normally nonsingular. This is the case if $M \to Y \leftarrow N$ are transverse.

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[4] Alain Connes and Georges Skandalis, *Théorème de l'indice pour les feuilletages*, C. R. Acad. Sci. Paris Sér. I Math. **292** (1981), no. 18, 871–876 (French, with English summary).MR623519 (82f:58079)

[5] ______, *The longitudinal index theorem for foliations*, Publ. Res. Inst. Math. Sci. **20** (1984), no. 6, 1139–1183.MR775126 (87h:58209)

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