Stability of Elliptic Harnack inequality

Mathav Murugan (UBC and PIMS)
Joint work with Martin Barlow

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Elliptic Harnack inequality (EHI)

- A. Harnack (1887): There exists $C > 1$ such that, for any ball $B(x,r) \subset \mathbb{R}^n$ and for any non-negative harmonic function $h : B(x,r) \to [0,\infty)$, we have

$$\sup_{B(x,r/2)} h \le C \inf_{B(x,r/2)} h.$$

- EHI is an easy consequence of Poisson integral formula

$$u(y) = \int_{\partial B(x,r)} u(\zeta) \frac{r^2 - |y-x|^2}{r\omega_{n-1} |y-\zeta|^n} \, d\zeta, \quad y \in B(x,r),$$

where ω_{n-1} is the surface area of $\mathbb{S}^{n-1}$.

- EHI implies the Liouville property: Every bounded below harmonic function on $\mathbb{R}^n$ is constant. (Proof: Replace h by $h - \inf_{\mathbb{R}^n} h$ and let $r \to \infty$ in EHI).

Parabolic Harnack inequality (PHI)

- Caloric function is a solution to the heat equation $\partial_t u = \Delta u$.
- PHI (J. Hadamard '54 and B. Pini '54): There exists $C > 1$ such that for all $x \in \mathbb{R}^n$, $r > 0$ and for all non-negative caloric function u defined on the time-space cylinder $(0, r^2) \times B(x, r)$ we have

$$\sup_{(r^2/4, r^2/2) \times B(x, r/2)} u \leq C \inf_{(3r^2/4, r^2) \times B(x, r/2)} u.$$

- PHI implies EHI (Proof: $u(t, \cdot) = h(\cdot), \forall t \in \mathbb{R}$ is caloric whenever h is harmonic).

Harnack inequalities are stable in $\mathbb{R}^n$

Jürgen Moser proved elliptic Harnack inequality (1961) and parabolic Harnack inequality (1964) for uniformly elliptic operators in divergence form

$$\mathcal{L}f(x) = \operatorname{div}(A(\cdot)\nabla f)(x) = \sum_{i,j=1}^{n} \partial_i(a^{ij}(\cdot)\partial_j f(\cdot))(x),$$

where the matrix $A(x) = [a^{ij}(x)]_{1\le i,j\le n}$ is measurable, essentially bounded, symmetric and comparable to the identity matrix in the following sense: there exists $K \ge 1$ such that

$$\frac{1}{K} I_n \le A(x) \le K I_n \quad \forall x \in \mathbb{R}^n.$$

Why study Harnack inequalities?

- Moser's motivation was to obtain Hölder continuity of weak solutions to PDE with uniformly elliptic operators.
- Hilbert's 19th problem: Are minimizers to $u \mapsto \int_\Omega F(\nabla u)\, dx$ smooth? Here $F : \mathbb{R}^n \to [0, \infty)$ is smooth, strictly convex and satisfies a growth condition and u has a prescribed boundary condition on $\partial\Omega$. (Dirichlet energy: the case $F(x) = \|x\|_2^2$)
- Positive answer given independently by E. De Giorgi (1957) and J. Nash (1958) by establishing Hölder continuity of solutions for uniformly elliptic operators.
- The De Giorgi-Nash-Moser estimates are of fundamental importance to quasi-linear elliptic and parabolic PDE.

PHI and heat kernel estimates

- Using Moser's PHI, D. Aronson ('68) proved two sided Gaussian estimates for the heat kernel associated with an uniformly elliptic operator. There exists $C, c > 0$ such that

$$\frac{c}{t^{n/2}} \exp\left(-\frac{d(x,y)^2}{ct}\right) \le p_t(x,y) \le \frac{C}{t^{n/2}} \exp\left(-\frac{d(x,y)^2}{Ct}\right),$$

  for all $x, y \in \mathbb{R}^n$ and for all $t > 0$.
- For general manifolds/graphs, we replace $t^{n/2}$ by $V(x, \sqrt{t})$.
- PHI is equivalent to the two sided Gaussian estimates on the heat kernel (Fabes and Stroock '86).
- Harnack inequality and its variants apply to non-local operators and to non-linear equations (e.g. Ricci flow).

Gradient Harnack inequality

- S.T. Yau ('75) proved Liouville property for manifolds with non-negative Ricci cuvature using a gradient estimate.
- S.Y. Cheng and S.T. Yau ('75) developed a local version of the gradient estimate for manifolds with bounded below Ricci curvature.
- Consider a manifold with non-negative Ricci curvature. Then there exists $C > 0$ such that for any ball $B(x, r)$ and for any positive harmonic function h on $B(x, r)$

$$|\nabla (\ln h)| \leq \frac{C}{r} \quad \text{in } B(x, r/2).$$

- P. Li and S.T. Yau ('86) proved a parabolic version of the above estimate that yields PHI for manifolds with non-negative Ricci curvature.

Gradient estimates vs Moser's Harnack inequalities

- The gradient estimates of S.T. Yau and his co-authors are stronger than Moser's Harnack inequalities and yield sharper constants.
- However, the assumption on curvature lower bound is not robust. To see this, let (M, g) is a manifold with non-negative Ricci curvature and $(M, \widehat{g})$ is another metric on M such that

$$K^{-1}g \le \widehat{g} \le Kg, \quad \text{for some } K > 1.$$

- Grigor'yan and Saloff-Coste ('92) independently showed that $(M, \widehat{g})$ satisfies PHI. However $(M, \widehat{g})$ need not have any curvature lower bound and the gradient Harnack inequality need not be true.

Which spaces satisfy PHI?

Theorem (A. Grigor'yan '91, L. Saloff-Coste '92)

The following are equivalent for a Riemannian manifold (M, g).

(a) *The conjunction of volume doubling property and Poincaré inequality: there exists $C_D > 0$ such that for all $x \in M, r > 0$*

$$V(x, 2r) \leq C_D V(x, r);$$

there exists $C_P > 0$ such that for all $f \in \mathcal{C}^\infty(M), x \in M, r > 0$

$$\int_{B(x,r)} \left| f - f_{B(x,r)} \right|^2 d\mu \leq C_P r^2 \int_{B(x,r)} \|\nabla f\|_2^2 \, d\mu.$$

(b) *The scale-invariant parabolic Harnack inequality holds for non-negative solutions of the heat equation $\partial_t u = \Delta_g u$.*

(c) *The transition probability density for the Brownian motion on (M, g) has Gaussian upper and lower bounds.*

Consequence : Stability under perturbations

- The parabolic Harnack inequality and Gaussian heat kernel bounds are stable under bounded perturbations of the Riemannian metric.
- Let (M, g) and $(\widehat{M}, \widehat{g})$ be quasi-isometric Riemannain manifolds. That is there exist $K \geq 1$ and a diffeomorphism $\phi : M \to \widehat{M}$ such that

  $$K^{-1} g(\xi, \xi) \leq \widehat{g}(d\phi(\xi), d\phi(\xi)) \leq K g(\xi, \xi), \quad \forall \xi \in TM.$$

  Then (M, g) satisfies PHI if and only if $(\widehat{M}, \widehat{g})$.
- Question: Which properties of a manifold are preserved under quasi-isometries?

Extensions to other settings

- The characterization of PHI has been extended to diffusions on local Dirichlet spaces (K.T. Sturm '95), random walks on graphs (T. Delmotte '99) and random walks on metric spaces (M., L. Saloff-Coste '15).
- A close variant of PHI holds for fractal and fractal-like spaces. A similar characterization was established by M. Barlow and R. Bass '03 for random walks on fractal-like graphs and by M. Barlow, R. Bass and T. Kumagai '06 for diffusions on fractals and fractal-like spaces.

Weighted graphs

- A weighted graph (G, μ) is a graph $G = (V, E)$ equipped with a conductance/weight $\mu : E \to (0, \infty)$.
- We shall always assume that the graph is bounded degree and the controlled weights: There exists $c > 0$ such that

$$\mathbb{P}(x, y) = \frac{\mu_{xy}}{\sum_{z \sim x} \mu_{xz}} \geq c, \quad \forall (x, y) \in E.$$

- The weighted Laplace operator

$$\Delta_\mu f(x) = \frac{1}{\sum_{y \sim x} \mu_{xy}} \sum_{y \sim x} \mu_{xy} \left(f(y) - f(x) \right).$$

- Question: Which properties of a weighted graph are preserved under bounded perturbation of conductances?
- $(G, \widehat{\mu})$ is a bounded perturbation of (G, μ) if there exists constant $K \geq 1$ such that $K^{-1}\mu \leq \widehat{\mu} \leq K\mu$.

Is EHI stable?

T. Lyons '87 showed that Liouville property is not preserved under quasi-isometries. I. Benjamani '91 provided a simpler counterexample.

$$\underset{\text{Stable}}{\text{PHI}} \implies \underset{?}{\text{EHI}} \implies \underset{\text{Not stable}}{\text{Liouville Property}}$$

However,

$$\text{Liouville Property} \not\Longrightarrow \text{EHI} \not\Longrightarrow \text{PHI}.$$

Question 1: Is EHI stable under quasi-isometries of a Riemannian manifold/bounded perturbation of conductances in a weighted graph?

Question 2: If so, characterize EHI by properties that are stable under bounded perturbations.

Answer 1: Yes, EHI is stable. (Barlow, M. '16)

Answer 2: We will get there in a few more slides.

EHI $\not\Longrightarrow$ PHI: Example 1

[Grigor'yan and Saloff-Coste '05] Define the measure

$$d\mu_\alpha(x) = (1 + |x|^2)^{\alpha/2}\, dx,$$

on $\mathbb{R}^n$, $n \geq 2$.

Consider the family of diffusions corresponding to the Dirichlet form

$$\mathcal{E}_\alpha(f, f) = \int |\nabla f|_2^2 \, d\mu_\alpha$$

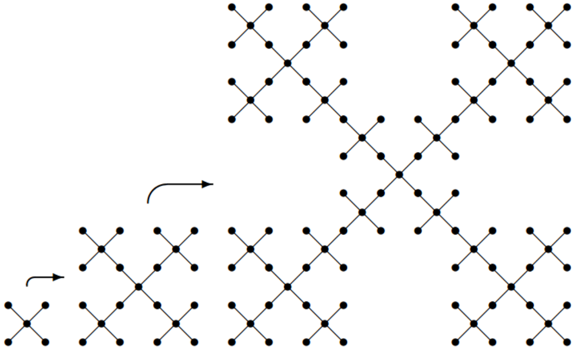
on $L^2(\mathbb{R}^n, \mu_\alpha)$ with generator (weighted Laplacian)

$$L_\alpha = \left(1 + |x|^2\right)^{-\alpha/2} \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(\left(1 + |x|^2\right)^{\alpha/2} \frac{\partial}{\partial x_i} \right) = \Delta + \alpha \frac{x.\nabla}{1 + |x|^2}.$$

This family of examples satisfy EHI for all $\alpha \in \mathbb{R}$ but satisfy PHI (volume doubling) if and only if $\alpha > -n$.

EHI $\not\Longrightarrow$ PHI: Example 2 – T. Delmotte '02

Consider any tree with polynomial volume growth r^{d_f} with $d_f > 1$. Glue it to the graph $\mathbb{Z}$ but connecting an arbitrary vertex in one graph to an arbitrary vertex in the other with an edge.



Viscek tree has $d_f = \log_3 5$ [Source: Isoperimetry, volume growth and random walks by C. Pittet and L. Saloff-Coste.]

EHI $\not\Longrightarrow$ PHI: Example 3 – Diffusion on fractals

Several classes of diffusions on fractals satisfy the following heat kernel estimates (transition probability of Brownian motion):
There exists $d_w > 1$ and $C, c > 0$ such that

$$p_t(x,y) \le \frac{C}{V(x, t^{1/d_w})} \exp\left(- \left(\frac{d(x,y)^{d_w}}{Ct} \right)^{1/(d_w - 1)} \right),$$

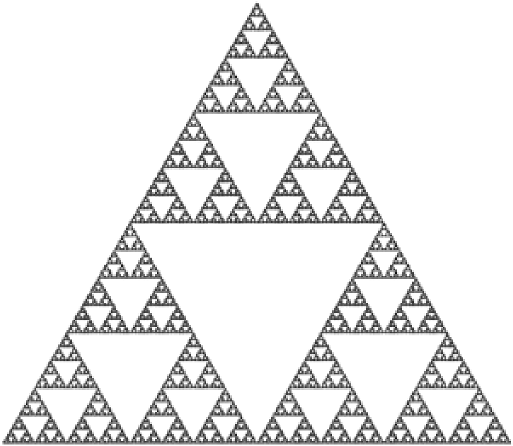
where the heat kernel p_t satisfies a matching lower bound with C replaced by c.
The parameter d_w governs the space-time scaling of the diffusion and called the exit time exponent since

$$\mathbb{E}_x \tau_{B(x,r)} \asymp r^{d_w}, \quad \forall x, \forall r > 0.$$

This generalization of Gaussian estimates are called sub-Gaussian estimates.

EHI $\not\Rightarrow$ PHI: Example 3 – Diffusion on fractals

Sierpinski gasket satisfies sub-Gaussian upper and lower bounds with $d_w = \log_2 5 \neq 2$ (M.T. Barlow and E. Perkins '88).



EHI $\not\Longrightarrow$ PHI: Example 3 – Diffusion on fractals

The sub-Gaussian estimate with escape time exponent d_w is equivalent a parabolic Harnack inequality PHI(Ψ) with a scaling given by the space-time scale function

$$\Psi(r) = r^{d_w}.$$

PHI(Ψ): There exists $C > 1$ such that for all balls $B(x, r)$ and for all non-negative caloric function u defined on the time-space cylinder $(0, \Psi(r)) \times B(x, r)$ we have

$$\sup_{(\Psi(r)/4, \Psi(r)/2) \times B(x, r/2)} u \leq C \inf_{(3\Psi(r)/4, \Psi(r)) \times B(x, r/2)} u.$$

By the same argument as before PHI(Ψ) implies EHI but PHI does not hold unless $d_w = 2$.

PHI(Ψ) is stable

Theorem (Barlow-Bass '03; Barlow-Bass-Kumagai '06)

The following are equivalent

(a) *Volume doubling property, a Poincaré inequality* PI(Ψ) *and a cutoff-Sobolev inequality* CS(Ψ).

(b) *A scale-invariant parabolic Harnack inequality* PHI(Ψ).

(c) *A two-sided sub-Gaussian estimate for the heat kernel corresponding to the space-time scale function* Ψ.

Stability of PHI(Ψ) follows the above theorem because (a) is stable under bounded perturbations.

Here Ψ satisfies the following condition: There exists $\beta_1, \beta_2 > 0$ such that

$$\left(\frac{R}{r}\right)^{\beta_1} \lesssim \frac{\Psi(R)}{\Psi(r)} \lesssim \left(\frac{R}{r}\right)^{\beta_2}, \quad \forall R > r > 0.$$

A fractal like manifold

Example of a manifold satisfying PHI(Ψ) with

$$\Psi(r) = r^{2}\mathbf{1}_{r\le 1} + r^{\log_2 5}\mathbf{1}_{r>1}.$$

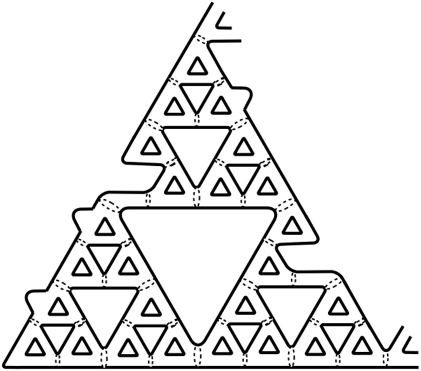


Image Source: Barlow, Bass and Kumagai

Energy measure

The energy measure $\Gamma(f,f)(\cdot)$ of a function $f \in \mathcal{F} \cap L^\infty$ for a Dirichlet form $(\mathcal{E},\mathcal{F})$ on $L^2(\mathcal{X},\mu)$ is defined as the unique Borel measure

$$\int_M g\, d\Gamma(f,f) = \mathcal{E}(f, fg) - \frac{1}{2}\mathcal{E}(f^2, g)$$

for all $g \in \mathcal{F} \cap C_c(\mathcal{X})$. Examples:
For Brownian motion on Riemannian manifold

$$d\Gamma(f,f)(x) = |\nabla f(x)|^2 \mu(dx), \quad \text{where } \mu = \text{Riemannian measure.}$$

For random walk on weighted graphs (G,μ), we have

$$\Gamma(f,f)(x) = \frac{1}{2}\sum_{y\sim x} (f(x)-f(y))^2 \mu_{xy}.$$

Poincaré inequality PI(Ψ)

Consider the Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(\mathcal{X}, \mu)$.
PI(Ψ): There exists $C_P > 0$ such that for all $f \in \mathcal{F}, x \in \mathcal{X}, r > 0$

$$\int_{B(x,r)} \left| f - f_{B(x,r)} \right|^2 \, d\mu \leq C_P \Psi(r) \int_{B(x,2r)} d\Gamma(f,f).$$

Cutoff Sobolev inequality CS(Ψ)

Definition (Cutoff function)

Let $U_1 \subset U_2$ be open sets in $\mathcal{X}$. We say that $\phi : \mathcal{X} \to \mathbb{R}$ is a cutoff function for $U_1 \subset U_2$ if $\phi \equiv 1$ on U_1 and $\phi \equiv 0$ on $U_2^{\complement}$.

Definition (Cutoff Sobolev inequality CS(Ψ))

We say that $(\mathcal{E}, \mathcal{F}, L^2(\mathcal{X}, \mu))$ satisfies the cutoff Sobolev inequality CS(Ψ) if there exists $C_1, C_2 > 0$ such that for all $x \in \mathcal{X}$, for all $r > 0$, there exists a cut-off function ϕ for $B(x,r) \subset B(x,2r)$ such that

$$\int_A f^2\, d\Gamma(\phi,\phi) \le C_1 \int_A d\Gamma(f,f) + \frac{C_2}{\Psi(r)} \int_A f^2\, d\mu \quad \forall f \in \mathcal{F} \cap L^\infty(\mathcal{X},\mu),$$

where $A = B(x,2r) \setminus B(x,r)$.

The cutoff Sobolev inequality CS(β) is stable under bounded perturbation of energy measure Γ and symmetric measure μ.

Characterization of EHI

Theorem (Barlow, M. '16)

Under mild assumptions on a Dirichlet space $(\mathcal{E}, \mathcal{F}, L^2(\mu))$ *on the metric space* $(\mathcal{X}, d)$*, the following are equivalent*

(a) *There exists a Radon measure* $\widehat{\mu}$ *satisfying the volume doubling property, a space-time scale function* $\Psi : \mathcal{X} \times (0, \infty) \to (0, \infty)$*,* $\beta_1, \beta_2 > 0$ *such that*

$$\left(\frac{R}{r}\right)^{\beta_1} \lesssim \frac{\Psi(x, R)}{\Psi(x, r)} \lesssim \left(\frac{R}{r}\right)^{\beta_2}, \quad \forall x \in \mathcal{X}, \forall R > r > 0,$$

and

$$\Psi(x, r) \asymp \Psi(y, r), \quad \forall x, y \in \mathcal{X}, \forall r > 0 \text{ such that } d(x, y) \leq r.$$

Furthermore, the corresponding time-changed Dirichlet space $(\mathcal{E}, \widehat{\mathcal{F}}, L^2(\widehat{\mu}))$ *satisfies* PI(Ψ)*and* CS(Ψ)*.*

(b) *Elliptic Harnack inequality.*

Poincaré inequality PI(Ψ)

PI(Ψ): There exists $C_P > 0$ such that for all $f \in \mathcal{F}, x \in \mathcal{X}, r > 0$

$$\int_{B(x,r)} \left| f - f_{B(x,r)} \right|^2 d\widehat{\mu} \leq C_P \Psi(x, r) \int_{B(x,2r)} d\Gamma(f, f).$$

Consequence: Stability of EHI

- Under mild hypotheses, EHI is a quasi-isometry invariant of weighted Riemannian manifolds.
- EHI is a invariant under bounded perturbation of conductance of a weighted graph.
- Under mild hypothesis, EHI is a rough isometry invariant. (rough isometry = quasi-isometry in the sense of Gromov)
- Generalized Moser's EHI: Under mild hypotheses, if a Riemannian manifold satisfies EHI for the Laplace-Beltrami operator, then it satisfies EHI for any uniformly elliptic operator on the Riemannian manifold.

Partial progress by Barlow '04, Bass '13

- M. Barlow '04: If a graph satisfies EHI, then there exists $C, \alpha > 0$ such that

$$|B(x,r)| \leq Cr^{\alpha}, \quad \forall x \in V, \forall r > 1.$$

- R. Bass '13 proved a characterization of EHI similar to our result, but he assumed volume doubling property for the symmetric measure along with certain regularity conditions on capacity that rule out Examples 1 and 2.
- On such assumptions, Bass remarks *"Since every known approach to proving an EHI uses volume doubling in an essential way, the problem of finding necessary and sufficient conditions for the EHI to hold without assuming any regularity looks very hard."*

Few remarks on the proof

- Study of diffusions on fractals provided fruitful ideas for the solution to a question about Riemannian manifolds/graphs.
- The proof of EHI from (a) follows the existing proofs in the literature that go back to work of De Giorgi/Moser.
- Probabilistic interpretation of $\Psi(x, r)$: Expected exit time for the time-changed process started at x to exit $B(x, r)$.
- Analytic interpretation of $\Psi(x, r)$: Inverse spectral gap of the 'Neumann Laplacian' on the ball $B(x, r)$ corresponding to the time changed process.

Few remarks on the proof

- Time change of the process (or equivalently change of symmetric measure of the Dirichlet form) does not affect the space of harmonic functions. Therefore one could hope to construct a nicer measure satisfies volume doubling along with Poincaré and cut-off Sobolev inequalities.
- A metric space $(\mathcal{X}, d)$ satifies metric doubling property, if there exists $M > 1$ such that, every ball of radius r can be covered by at most M balls of radius $r/2$, for all $r > 0$.
- A metric space $(\mathcal{X}, d)$ admits a measure that satisfies volume doubling property if and only if it satisfies the metric doubling property. (A. Vol'berg, S.V. Konyagin '87, J. Luukkainen, E. Saksman '98).
- The proof of existence of doubling measure is not constructive.

Few remarks on the proof

- First step is to show that every space satisfying EHI satisfies the metric doubling property.
- This implies that the space admits a doubling measure. However, an arbitrary doubling measure need not satisfy the desired Poincaré and cut-off Sobolev inequalties.
- We use the ideas of A. Vol'berg, S.V. Konyagin, J. Luukkainen, E. Saksman with some new ingredients to "construct" the desired measure satisfying volume doubling along with Poincaré and cut-off Sobolev inequalities.

Thank you