Stochastic Parameterisation Schemes Based on Rigorous Limit Theorems

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Stochastic climate modeling

Most stochastic climate models are specific to the modeled system, but system-specific SDE models in particular implicitly apply limit theorems, which give general formulas.
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\[ \frac{dX}{dt} = LX, \]

→ the equations of motion are linearised about a mean state, white noise is added to account for the (fast-evolving) error in linearisation, and a damping term is added for stability:

\[ \frac{dX}{dt} = (L + D)X + \frac{dW}{dt} \]
Theorem-based reduction methods

Hasselmann(1976)
Papanicolaou(1976)

Khasminskii(1966)


Fatkullin and Vanden-Eijnden(2004)

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\begin{align*}
\frac{dx}{dt} &= f(x, y) \quad \text{(slow climate mode)} \\
\frac{dy}{dt} &= \frac{1}{\epsilon} g(x, y) \quad \text{(fast weather mode)}
\end{align*}
\]

As \( \epsilon \to 0 \), \( x \to X \) in distribution, where \( X \) satisfies:

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\frac{dX}{dt} = \bar{f}(X) + \epsilon D(X) + \sqrt{\epsilon} \sigma(X) \frac{dW}{dt}
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\* Simple to implement: do not have to resolve fast mode, \( y \)
Online closure

Figure 3: Schematic illustration of the projective integration scheme
\[ \frac{dx}{d\tau} = \frac{1}{\epsilon} f_1(x, y) + f_0(x, x) \]
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\[ \to \quad \text{In particular,} \quad D = \int_0^\infty \mathbb{E}(\nabla f_1(x, y_t))(f_1(x, y_t), g_1(x, y_t))^* \, dt \]
Franzke et al. (2005)

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Tests of assumptions

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  - ergodicity and mixing of fast dynamics
  - the existence of limiting slow dynamics
Atmospheric low-frequency variability (LFV)
QG model of LFV (Kravtsov et al. (2005))
Bifurcation: unimodal to bimodal distribution of jet axis
Existence of limiting slow dynamics

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\frac{dx}{dt} = f(x, y)
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- Speeding up the fast mode is equivalent to changing the bifurcation parameter (the bottom drag parameter).
- In large neighbourhood of bifurcation point, not every set of slow variables has limiting slow dynamics.
EOFs in region of jet bimodality

Autocorrelation timescale and % explained variance vs. EOF; \( k^{-1} = 6.7 \text{ days} \)

- wave-4
- stationary mode
Schematic of KRG05 dynamics

- Bifurcation point
- Unimodal jet distribution
- Bimodal jet distribution
- Increasing spin-down timescale
Method of Franzke et al. (2005); 1-D SDE

Stationary mode PDFs of unreduced and 1-D regressed reduced models;

$k^{-1} = 6.7$ days

Principal component

unreduced model
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Method of Franzke et al. (2005); 1-D SDE

Stationary mode time series of regressed reduced models w/out wave-4; $k^{-1} = 6.7$ days

$\lambda_B = \lambda_A = \lambda_F = 0.175$

$\lambda_B = \lambda_A = \lambda_F = 0.1$
Method of Franzke et al. (2005); 3-D SDE

Time series of stationary mode; $k^{-1} = 2.3$ days
Regressed reduced model with $\lambda_B = \lambda_A = \lambda_F = 0$

Regressed reduced model with $\lambda_B = \lambda_A = \lambda_F = 0.15$
Hasselmann’s method

Stationary mode PDFs of unreduced and reduced models; $k^{-1} = 6.7$ days
Deterministic averaging–DNS hybrid; 4X faster than DNS

Unreduced model
Reduced model

Deterministic averaging with additive white noise

Unreduced model
Reduced model
Conclusions

- In a large neighbourhood of bifurcation point there is no limiting 3-D slow dynamics despite good scale separation, highlighting the importance of tests of existence of limits.
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  → Limiting slow dynamics can differ considerably from slow dynamics, even for order of magnitude scale separation.

- The conclusion of KRG05 that first-order dynamics of jet bimodality arises from interaction between stationary and wave-4 modes is incorrect.

  → Leading fast synoptic eddies are of first-order importance and wave-4 facilitates transitions between states.
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■ For larger models, off-line calculations would be impractical without further simplifications
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