NONHOLONOMIC SYSTEMS AND THE HAMILTONIZATION PROBLEM.

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Abstract

A nonholonomic system is a mechanical system with constraints in the velocities and thus the equations of motion are not hamiltonian. This fact has consequences in the dynamical behaviour as well as in the geometry underlying the mechanical system. In this talk we will study geometric features of nonholonomic systems and their behaviour after a reduction by a group of symmetries. In particular, we are interested in how Poisson geometry may help in understanding how far these systems are from being hamiltonian.

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ON THE LOCATION OF SATURN F RING: A SIMPLE MODEL

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Abstract

The confinement of planetary narrow rings is understood in terms of the shepherd theory [1]. This theory proposes the existence of two shepherd moons that orbit around the ring, and involves angular momentum transfer mechanisms between them and the ring particles, self-gravity, viscous damping, and assumes the existence of lower-order resonance at the ring boundaries in order to confine the ring. Saturn's F ring is a fascinating example of a narrow eccentric ring that displays a rich dynamical structure: besides its non-zero eccentricity and sharp-edges, it has multiple components entangled in a complicated way which shows a variety of short-time features [2]. The two shepherd moons, Prometheus and Pandora, influence importantly many of the short-time dynamical features observed. Yet, the shepherd theory does not apply to this ring since there are no mean-motion resonances that confine the ring and the mass of the shepherd moons is too small [3]. The confinement of Saturn's F-ring thus remains unexplained.
In this talk, I shall describe our scattering approach to narrow rings [4], illustrating it on a simple billiard system, where the relevant physics can be easily understood. Within this approach, I shall present numerical results based of accurate long-time integrations on a realistic 5-body model for the occurrence and location of Saturn's F ring. Test particles that remain trapped and display some stability properties form a narrow elliptic ring displaying sharp edges. Comparison of our results with the observations shall be provided. This work is in collaboration with Àngel Jorba (U. Barcelona).

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VORTEX DYNAMICS ON SURFACES OF REVOLUTION

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Abstract

Many key features of the motion of satellites, planets, stars, even galaxies can be captured by point mass dynamics. Likewise, many key features of fluid motion such as atmospheric storms, ocean eddies, super fluid vortices, and early stages of mass aggregation in gravitational systems can be captured by point vortex dynamics. Yet, serious mathematical challenges remain. Systems consisting of more than a few points are non-integrable, and complexity increases dramatically with the number of particles. Furthermore, the underlying geometry has a profound influence on the particle motion, as has only just begun to be investigated. Indeed one of today' s challenges is a formulation of the N-vortex dynamics on Riemann surfaces. There are formulations over surfaces with constant Gaussian curvature [1, 4, 5, 6, 9, 10], and lately, for surfaces with not constant Gaussian curvature, conform to the sphere [2, 7]. We present some results about vortex dynamics on surfaces of revolution (among others the sphere and the ellipsoid of rotation) and we show the importance of the curvature in the stability of the relative equilibria

[1, 2]. The stabilization by curvature appears also in the classical N-body problem [5, 8]

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STABILITY RESULTS OF SOME MODELS IN CELESTIAL MECHANICS

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Abstract

I will present some stability results based on KAM theory. I will consider both conservative and dissipative (i.e., conformally symplectic) systems (either maps and flows). The proof is constructive and it provides efficient algorithms to evaluate the breakdown threshold of invariant tori ([2]). For low–dimensional degrees of freedom, KAM theory allows to get stability results. In this context, I will discuss the stability of the restricted 3–body problem (with particular reference to the asteroid 12 Victoria) and the stability of the Moon–Earth synchronous resonance under tidal effects. I will show that computer–assisted proofs may be devised in order to show the existence of invariant tori for realistic values of the parameters ([1], [3], [4]).

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CLOSED TRIPLE COLLISION ORBITS IN PHOTOIONIZATION OF THE HELIUM ATOM

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Abstract

The cross section for single-electron photoionization in the helium atom shows fluctuations which decrease in amplitude as the total energy of the atom approaches the double-ionization threshold [1]. Treating the helium atom as a three-body Coulomb problem, we predicted that the decay of the fluctuations can be characterized in terms of a threshold law with an exponent obtained as a combination of Siegel's stability exponents of the triple-collision singularity; the details of the fluctuations were predicted to be linked to a set of infinitely unstable classical orbits starting and ending at the nonregularizable triple collision - so called CTCOs (Closed Triple Collision Orbits) [2]. The validity of the prediction was confirmed by numerical calculations for the *collinear* helium model [3], which was presented at the workshop 'Few-Body Dynamics in Atoms, Molecules, and Planetary Systems' held in 2010. We extend our calculational method to planar helium and analyze the results to confirm the validity of the prediction for the *planar* helium model atom. In addition, it is shown that the CTCOs play the same role in the partial cross sections as well as in the total cross section.

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RELATIVE EQUILIBRIA OF 3 BODIES IN DIMENSION d

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Abstract

The 3-body problem can be effectively symmetry reduced using invariants [1]. This leads to a Lie-Poisson structure and we analyse the equilibria of this 10-dimensional reduced system with two Casimirs. The Casimirs are the length of the angular momentum vector, and the Gram determinant of four difference vectors. When the spatial dimension d is equal to three then the Gram determinant is zero and the classical relative equilibria of Euler and Lagrange are found. However, when $d > 3$ the Gram determinant may be non-zero, and this allows for additional families of relative equilibria, with shapes that are distinct from the collinear and the equilateral families. These additional relative equilibria where described by Albouy and Chenciner [2] who called them balanced configurations [3]. We will give a detailed description of such relative equilibria and their linear stability.

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TRAIN CORRESPONDENCES FOR CELESTIAL BODIES

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Abstract

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By kinematics of scattering we mean motion of n particles on line segments of $\mathbb{R}^d$, with two-body collisions at the end points, preserving momentum and energy. It is known (see [1]) that the number of collisions is finite.

We analyze the kinematics of scattering. Then, adapting techniques from [2], we show that every transversal kinematical scattering process can be approximated by the dynamics of particles with Coulomb interaction (attractive or repulsive), in the limit of large energy or large spatial extensiveness.

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QUANTUM AND CLASSICAL MOTION OF TWO ELECTRONS IN ATOMS AND LASER FIELDS

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Abstract

In atomic physics, the few-body problem arises already for the second element in the periodic table, helium (He), exhibiting two electrons "orbiting" inside the $1/r$ Coulomb potential around the doubly-charged nucleus. Any atom or ion besides hydrogen (H) or H-like systems is thus in general not analytically solved and theoretical description relies on approximation or numerics that need to be tested.

Here, experiments are presented on measurements and the control of two-electron dynamics in atoms by applying laser/electric fields, on time scales of femtoseconds to attoseconds. Quantum-mechanical correlated wave packets of two excited electrons in helium can be observed and controlled [1]. In stronger laser fields, quasi-classical recollision occurs, where a single electron is freed, accelerated in the laser field and returns to its parent ion where it can further excite other electrons. Experiments show that a two-electron (doubly-) excited state can also be formed in this process [2]. Comparing measurements with classical trajectory simulations, we find that two electrons are then removed from the atom sequentially, but within a fraction of a laser-optical cycle ($\sim$200 as) even when initially bound to a fully symmetric orbit.

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BIFURCATION ANALYSIS OF SYMMETRIC HAMILTONIAN RESONANCES

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Abstract

We present a general analysis of the bifurcation sequences of periodic orbits in general position of resonant Hamiltonian normal forms invariant under $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry. The low-order cases 2:2 and 2:4 show rich structure [1, 2, 3] investigated both with geometric methods and a singularity theory approach.

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STABILITY OF RELATIVE EQUILIBRIA IN THE PLANAR N-VORTEX PROBLEM: A TOPOLOGICAL APPROACH

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Abstract

In the weather research and forecasting models of certain hurricanes [1], vortex crystals are found within a polygonal-shaped eyewall. These special configurations can be interpreted as relative equilibria (rigidly rotating solutions) of the point vortex problem introduced by Helmholtz. Their stability is thus of considerable importance. Adapting the approach of Moeckel [2] for the companion problem in celestial mechanics, we present some theory and results on the linear and nonlinear stability of relative equilibria in the planar N-vortex problem [3]. A topological approach is taken to show that for the case of positive circulations, a relative equilibrium is linearly stable if and only if it is a nondegenerate minimum of the Hamiltonian restricted to a level surface of the angular impulse (moment of inertia). Using a criterion of Dirichlet's, this implies that any linearly stable relative equilibrium with positive vorticities is also nonlinearly stable. Two symmetric families, the rhombus and the isosceles trapezoid, are analyzed, with stable solutions found in each case.

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REGULARIZATION OF THE KEPLER PROBLEM ON THE SPHERE

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Abstract

In this talk I will present some results concerning the Kepler problem on the three-sphere S^3, that I obtained in collaboration with Shengda Hu [1]. I will show how to perform a Moser-type regularization and how to adapt the Ligon-Schaaf regularization to the problem under consideration. Then, I will explain the relationship between these regularizations and the corresponding regularizations for the Kepler problem in $\mathbb{R}^3$. This will be done by showing that the Moser regularization and the Ligon-Schaaf map we obtained can be understood as the composition of the corresponding maps for the Kepler problem in Euclidean space, and the gnomonic transformation.

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NORMAL FORMS FOR ROTATIONALLY-INVARIANT SYSTEMS

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Abstract

Rotationally invariant cotangent-bundle systems and the particular the case of n-mass point systems are studied using a recently-obtained parametrisation of the phase space. This parametrisation, in which the Marsden-Weinstein symplectic reduced space becomes a vector space with a canonical symplectic form, is beneficial in the study of dynamics near relative equilibria. In particular, it can be shown that Nekhoroshev's estimates near an elliptic equilibrium point in the reduced space induce similar estimates in the full phase-space.

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INVARIANT TORI IN THE SPATIAL 3-BODY PROBLEM BY AVERAGING AND REDUCTION

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Abstract

We study the spatial three-body problem, which is a system of nine degrees of freedom. In order to apply perturbation theory we need to establish the possible regimes where the Hamiltonian of the three-body problem can be split into two parts: the unperturbed Hamiltonian composed of two Keplerian terms and the perturbation, which is supposed to be small with respect to the principal part.

Our approach is based on a combination of averaging techniques with reduction theory with the aim of building a reduced Hamiltonian and a reduced phase space as simple as possible. The reduction process takes into account all possible continuous symmetries of the problem, including the symmetry generated by the two approximate integrals obtained after performing the normalisation with respect to the two fast angles and truncating the higher-order terms.

Once the reduction process is completed, the averaged Hamiltonian defines a system of one degree of freedom in the fully reduced space whose dimension is two and which is embedded in $\mathbb{R}^3$. This phase space is a surface that depends on three parameters, it is parametrised using Deprit's elements [1] and may have zero, one, two or three singular points. Our approach is global in the sense that we deal with the flow of the fully-reduced system in the whole fully-reduced space.

The next step is the discussion of the occurrence of the different relative equilibria of the reduced Hamiltonian system. This is done in terms of the invariants and the fundamental constraint that define the fully-reduced phase space. There are two basic parameters to perform the analysis, namely the modulus of the total angular momentum vector and the modulus of the angular momentum vector of the outer ellipse. They generate the plane of parameters which is divided into six different regions and presents five bifurcation lines. Each region has a different number of relative equilibria, ranging from two to six. The number and stability of the equilibria change when crossing the different lines [3].

We consider every elliptic point in the reduced phase space and reconstruct it step by step to compute the corresponding KAM invariant tori that persist in the original space $\mathbb{R}^{12}$. In the cases where the equilibria are placed on the singular points of the two-dimensional reduced phase space we need to make a local analysis around the singularities, passing to a surface with no singular points.

In the process of reconstruction we define intermediate spaces and sets of variables depending on the type of motions under consideration. Due to the scale in the perturbation, the Hamiltonian is too degenerate to apply classical results of KAM theory. Thus, we use a specific result by Han, Li and Yi [2] for this kind of systems.

We find Cantor families of five-dimensional KAM tori for the spatial three-body problem.

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RECOLLIDING PERIODIC ORBITS IN ATTOSECOND PHYSICS

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Abstract

Can an ionized electron be driven back to the core by an ultrastrong laser pulse? This is a high-stakes issue in attosecond physics since the returning electron, by carrying back the energy it has absorbed from the laser, can act as the agent of many key processes in intense laser physics, including the ultrafast imaging of macromolecules and the design of new light sources through generation of ultra-high harmonics [1, 2]. When the laser is linearly polarized, the "recollision" (or "rescattering") model [3, 4] has been immensely successful in both interpreting current experiments and devising new ones. However the picture is much less clear for all other polarizations [5, 6]. The very same recollision model predicts that returns must be suppressed in circularly polarized laser pulses because ionized electrons tend to spiral away from the core. In contrast, we present evidence that electrons return to the core along specific families of periodic orbits which shuttle electrons towards and away from the core. We show that these special periodic orbits are the key agents of attosecond processes in circularly polarized fields.

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