A THEORY OF HYDRODYNAMIC TURBULENCE BASED ON NON-EQUILIBRIUM STAT. MECH.

David Ruelle

IHES

Bowen Conference / 3 Aug 2017 / UBC
Overview.

- Study intermittency exponents $\zeta_p$ such that

$$\langle |\Delta v|^p \rangle \sim \ell^{\zeta_p}$$

where $\Delta v$ is contribution to fluid velocity at small scale $\ell$.

[ Claim: $\zeta_p = \frac{p}{3} - \frac{1}{\ln \kappa} \ln \Gamma\left(\frac{p}{3} + 1\right)$ ]

experimentally $(\ln \kappa)^{-1} = 0.32$, i.e., $\kappa \approx 20$ or $25$.

- Distribution of radial velocity increment and relation with Kolmogorov-Obukhov.

- Reynolds number $\approx 100$ at onset of turbulence.
References:


1. Obtaining the basic probability distribution.

- Kinetic energy goes down from large spatial scale $\ell_0$ to small scales through a cascade of eddies of increasing order $n$ so that

$$\mathbf{v} = \sum_{n \geq 0} \mathbf{v}_n$$

with viscous cutoff.

Eddy of order $n - 1$ in ball $R_{(n-1)i}$ decomposes after time $T_{(n-1)i}$ into eddies of order $n$ contained in balls $R_{nj} \subset R_{(n-1)i}$.

Balls $R_{nj}$ form a partition of 3-space into roughly spherical polyhedra of linear size $\ell_{nj}$, lifetime $T_{nj}$. 
• Assume that the dynamics of each eddy is universal, up to scaling of space and time, and independent of other eddies.

Conservation of kinetic energy $E$ yields

$$\sum_j \frac{E(R_{nj})}{T_{nj}} = \frac{E(R_{(n-1)i})}{T_{(n-1)i}}$$

Universality of dynamics and inviscid scaling give for initial eddy velocities

$$\frac{v_n}{\ell_{nj}} = \frac{T_{(n-1)i}}{T_{nj}} \cdot \frac{v_{n-1}}{\ell_{n-1}}$$

hence

$$\sum_j \int_{R_{nj}} \frac{|v_n|^3}{\ell_{nj}} = \int_{R_{(n-1)i}} \frac{|v_{n-1}|^3}{\ell_{(n-1)i}}$$

(implies intermittency).
• For simplicity assume size $\ell_{nj}$ depends only on $n$: $\ell_{(n-1)i}/\ell_{nj} = \kappa$. Then

$$\kappa \sum j \int_{R_{nj}} |\mathbf{v}_n|^3 = \int_{R_{(n-1)i}} |\mathbf{v}_{n-1}|^3$$

• Assume that the distribution of the $\mathbf{v}_n$ between different $R_{nj}$ maximizes entropy: microcanonical distribution $\rightarrow$ canonical distribution:

$$\sim \exp[-\beta |\mathbf{v}_n|^3] d^3 \mathbf{v}_n$$

Integrating over angular variables:

$$\sim \exp[-\beta |\mathbf{v}_n|^3] |\mathbf{v}_n|^2 d|\mathbf{v}_n| = \frac{1}{3} \exp[-\beta |\mathbf{v}_n|^3] d|\mathbf{v}_n|^3$$

hence $V_n = |\mathbf{v}|^3$ has distribution

$$\beta \exp[-\beta V_n] dV_n$$
Finally since the average value $\beta^{-1}$ of $V_n$ is $V_{n-1}/\kappa$, $V_n$ is distributed according to

$$\frac{\kappa}{V_{n-1}} \exp \left[ - \frac{\kappa V_n}{V_{n-1}} \right] dV_n$$

Starting from a given value of $V_0$ the distribution of $V_n$ is given by

$$\frac{\kappa}{V_0} \frac{dV_1}{V_0} e^{-\kappa V_1/V_0} \ldots \frac{\kappa}{V_{n-1}} \frac{dV_n}{V_{n-1}} e^{-\kappa V_n/V_{n-1}} \quad (\ast)$$

The validity of (\ast) is limited by dissipation due to the viscosity $\nu$: we must have

$$V_n^{1/3} \ell_n > \nu$$
2. Calculating $\zeta_p$.

- To compute the mean value of $|v_n|^p = V_n^{p/3}$ we note that

$$\frac{\kappa}{V_{n-1}} \int \exp \left[ -\frac{\kappa V_n}{V_{n-1}} \right] V_n^{p/3} dV_n = \left( \frac{V_{n-1}}{\kappa} \right)^{p/3} \int \exp[-w] w^{p/3} dw$$

$$= \kappa^{-p/3} V_{n-1}^{p/3} \Gamma \left( \frac{p}{3} + 1 \right)$$

hence, using induction and $\ell_n/\ell_0 = \kappa^{-n}$,

$$\langle V_n^{p/3} \rangle = \frac{\kappa}{V_0} \int \exp \left[ -\frac{\kappa V_1}{V_0} \right] dV_1 \cdots \frac{\kappa}{V_{n-1}} \int \exp \left[ -\frac{\kappa V_n}{V_{n-1}} \right] V_n^{p/3} dV_n$$

$$= \kappa^{-np/3} V_0^{p/3} \Gamma \left( \frac{p}{3} + 1 \right)^n = V_0^{p/3} \left( \frac{\ell_n}{\ell_0} \right)^{p/3} \Gamma \left( \frac{p}{3} + 1 \right)^n$$
Therefore

\[
\ln \langle |v_n|^p \rangle = \ln \langle V_n^{p/3} \rangle = \ln V_0^{p/3} + \frac{p}{3} \ln \left( \frac{\ell_n}{\ell_0} \right) - \frac{\ln(\ell_n/\ell_0)}{\ln \kappa} \ln \Gamma \left( \frac{p}{3} + 1 \right)
\]

\[
= \ln V_0^{p/3} + \ln \left( \frac{\ell_n}{\ell_0} \right) \cdot \left[ \frac{p}{3} - \frac{1}{\ln \kappa} \ln \Gamma \left( \frac{p}{3} + 1 \right) \right] = \ln \left[ V_0^{p/3} \left( \frac{\ell_n}{\ell_0} \right)^{\zeta_p} \right]
\]

where

\[\zeta_p = \frac{p}{3} - \frac{1}{\ln \kappa} \ln \Gamma \left( \frac{p}{3} + 1 \right)\]

or

\[
\langle |v_n|^p \rangle = V_0^{p/3} \left( \frac{\ell_n}{\ell_0} \right)^{\zeta_p} \sim \ell_n^{\zeta_p}
\]

as announced.

- If $r \approx \ell_n$ we have $u \approx u_n \approx$ radial component of $v_n$

  \[ F(u) = \left( \prod_{k=1}^{n} \int_{0}^{\infty} \frac{\kappa \, dV_k}{V_{k-1}} e^{-\kappa V_k / V_{k-1}} \right) \frac{1}{2V_n^{1/3}} \chi[-V_n^{1/3}, V_n^{1/3}](u) \]

  \[ = \frac{1}{2} \left( \frac{\kappa_n}{V_0} \right)^{1/3} \int \cdots \int_{w_1 \cdots w_n > (\kappa_n / V_0) |u|^3} \prod_{k=1}^{n} \frac{dw_k e^{-w_k}}{w_k^{1/3}} \]

- The distribution $G_n(y)$ of $y = (\kappa_n / V_0)^{1/3}|u|$ is given by

  \[ G_n(y) = \int \cdots \int_{w_1 \cdots w_n > y^3} \prod_{k=1}^{n} \frac{dw_k e^{-w_k}}{w_k^{1/3}} \]
• This satisfies
\[ e^t G_n(e^t) = (\phi^{* (n-1)} \ast \psi)(t) \] (**) 
with
\[ \phi(t) = 3 \exp(3t - e^{3t}), \quad \psi(t) = e^t \int_t^\infty e^{-s} \phi(s) \, ds \]

\[ \Rightarrow G_n(y) \text{ is a decreasing function of } y. \]

• For small \( u \), \( G_n \) gives a good description of the distribution of \( u \), with normalized \( \langle |u|^2 \rangle \) (see Schumacher et al.).

• (**) suggests a lognormal distribution with respect to \( u \) in agreement with Kolmogorov-Obukhov, but this fails because \( \phi, \psi \) tend to 0 only exponentially at \( -\infty \).
4. The onset of turbulence.

- We may estimate the Reynolds number $Re = \frac{|v_0|\ell_0}{\nu}$ for the onset of turbulence by taking

$$1 \approx \left\langle \frac{\nu}{|v_1|\ell_1} \right\rangle = \left\langle \frac{\nu}{V_1^{1/3} \kappa V_0} \right\rangle = Re^{-1} \left\langle \kappa^{4/3} \left( \frac{V_0}{\kappa V_1} \right)^{1/3} \right\rangle$$

[Relation to dissipation is dictated by dimensional arguments] \Rightarrow

$$Re \approx \kappa^{4/3} \int_{0}^{\infty} \left( \frac{\kappa V_1}{V_0} \right)^{-1/3} \frac{\kappa}{V_0} dV_1 e^{-\kappa V_1/V_0}$$

$$= \kappa^{4/3} \int_{0}^{\infty} \alpha^{-1/3} d\alpha e^{-\alpha} = \kappa^{4/3} \Gamma \left( \frac{2}{3} \right)$$

Taking $1/\ln \kappa = .32$ hence $\kappa^{4/3} = 64.5$, with $\Gamma(2/3) \approx 1.354$ gives $Re \approx 87$ agreeing with $Re \approx 100$ as found in Schumacher et al.