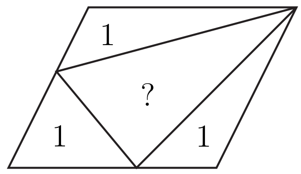
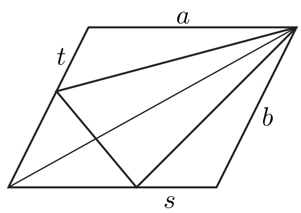
Solutions to May 2006 Problems

Problem 1. In the figure below, a parallelogram has been divided into four triangles. If the areas of the three "outer" triangles are each 1 as shown, what is the area of the fourth triangle?



Solution. At first sight it seems that the given data are not enough to determine the unknown area. But we will set up some equations and play around to see what happens. Let the dimensions of the triangles and parallelogram be as shown in the diagram below.



Let the area of the parallelogram be $2K$. Draw the diagonal of the parallelogram as shown. That diagonal divides the parallelogram into two triangles of area K.

Look first at the triangle on the lower right. By comparing with the area of the half-parallelogram, we can see that it has area $(s/a)K$. Similarly, the triangle on the uppper left has area $(t/b)K$. Both of these triangles have area 1. It follows that $a = sK$ and $b = tK$.

The triangle on the lower left has area $((a-s)/a)((b-t)/b)K$. From the fact that $a = sK$ and $b = tK$, the area simplifies to $((K-1)^2/K^2)K$. But this area is equal to 1. We therefore get $(K-1)^2 = K$, or equivalently $K^2 - 3K + 1 = 0$.

The quadratic equation has roots $(3 \pm \sqrt{5})/2$. One root is obviously too small, so $K = (3+\sqrt{5})/2$, and therefore the area of the parallelogram is $3+\sqrt{5}$. To find the area of the inner triangle, subtract the sum of the areas of the three outer triangles. We conclude that the inner triangle has area $\sqrt{5}$.

Comment. Suppose that the triangle at the lower left has area p, and the other two corner triangles have area q and r. The same argument as the one given above shows that the parallelogram has area

$$p + q + r + \sqrt{(p+q+r)^2 - 4qr}$$

and therefore the inner triangle has area $\sqrt{(p+q+r)^2 - 4qr}$.

Problem 2. Find the last non-zero digit of 1000!.

Problem 3. (a) Find numbers A, B, and C such that

$$\frac{1}{x(x+1)(x+2)} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x+2}$$

for all $x \neq 0$, -1, or -2.

(b) Simplify:

$$\frac{1}{1\cdot 2\cdot 3} + \frac{1}{2\cdot 3\cdot 4} + \frac{1}{3\cdot 4\cdot 5} + \cdots + \frac{1}{99\cdot 100\cdot 101}.$$

Problem 4. Find all pairs (x, y) of real numbers that satisfy the two equations

$$x + y = 1$$
$$x^5 + y^5 = 11.$$

Solution. We explore for a while a "standard" approach. From the first equation, we find that $y = 1 - x$. Substitute $1 - x$ for y in the second equation. We get $x^5 + (1-x)^5 = 11$. Expand $(1-x)^5$, either by using the Binomial Theorem or by patiently multiplying out. After some simplification we arrive at $x^4 - 2x^3 + 2x^2 - x - 2 = 0$. This equation has no obvious solution, maybe.

The equation is of degree 4 in x. There *is* a method for solving fourth-degree equations, one that goes back to the sixteenth-century mathematicians Cardano and Ferrari. But the method is complicated, and for now we do not pursue this approach any further.

It is better to exploit the symmetry in the problem, either by keeping the variables x and y, or by making a symmetry-preserving substitution. So if we are going to substitute, it seems wiser to note that if $x+y = 1$, then $x = 1/2+t$ and $y = 1/2 - t$ for some real number t. Substitute in the second equation. We get

$$(1/2 + t)^5 + (1/2 - t)^5 = 11.$$

Expand. There is substantial cancellation, and after a while we arrive at

$$16t^4 + 8t^2 - 35 = 0.$$

This is a quadratic equation in t^2. To put it another way, if we let $w = t^2$ we get a quadratic equation in w.

The quadratic equation can be solved for t^2 by using the Quadratic Formula. We get

$$t^2 = \frac{-8 \pm \sqrt{64 + 2240}}{32}.$$

But t is real, so t^2 can not be negative, and after some calculation we arrive at

$$t^2 = \frac{5}{4},$$

and conclude that $t = \pm\dfrac{\sqrt{5}}{2}$. Alternately, we could observe that

$$16t^4 + 8t^2 - 35 = (4t^2 - 5)(4t^2 + 7)$$

and therefore $t^2 = 5/4$.

Finally, we can write down the real solutions of the original system of equations. They are

$$(x, y) = \left(\frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}\right) \quad \text{and} \quad (x, y) = \left(\frac{1-\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}\right).$$

Comment. Of course we cheated in setting up the problem. The "11" in the equation $x^5 + y^5 = 11$ was chosen to make numbers turn out nicely, but not *too* nicely. However, even if we use a number other than 11, the argument does not change much. If for example we are given the equations $x + y = a$, $x^5 + y^5 = b$, we let $x = a/2 + t$; thus $y = a/2 - t$. When we substitute into $x^5 + y^5 = b$ and simplify, we obtain an equation which is quadratic in t^2. The numerical details change, but the structure does not.

Early in the proof, we remarked that $x^4 - 2x^3 + 2x^2 - x - 2 = 0$ has no obvious solution. That is not quite true. We *might* notice that the equation can be rewritten as $(x^2 - x - 1)(x^2 - x + 2) = 0$, and then everything is easy. If, however, we start with $x^5 + y^5 = b$, factorization is not at all obvious.

Another Way. We can keep both variables around for a while. There are many ways to do this. For example, we could use the fact that

$$x^5 + y^5 = (x + y)(x^4 - x^3y + x^2y^2 - xy^3 + y^4)$$

and therefore $x^4 - x^3y + x^2y^2 - xy^3 + y^4 = 11$.

But $x^2 + y^2 = (x + y)^2 - 2xy$. Let $p = xy$. Then $x^2 + y^2 = 1 - 2p$. Note now that $x^4 + y^4 = (x^2 + y^2)^2 - 2x^2y^2 = (1 - 2p)^2 - 2p^2$. Also, $x^3y + xy^3 = xy(x^2 + y^2) = p(1 - 2p)$. We therefore arrive at the equation

$$[(1 - 2p)^2 - 2p^2] + p^2 - [p(1 - 2p)] = 11.$$

After a little manipulation we arrive at $5p^2 - 5p - 10 = 0$. This quadratic equation has the roots $p = 2$ and $p = -1$. So we have the two possibilities $x + y = 1$, $xy = 2$, and $x + y = 1$, $xy = -1$.

Note that $(x - y)^2 = (x + y)^2 - 4xy = 1 - 4p$. This shows that $p = 2$ is impossible, since it would give $(x - y)^2 = -7$. If $p = -1$, we get $(x - y)^2 = 1 - 4p = 5$. That gives $x - y = \pm\sqrt{5}$. Finally, using $x + y = 1$ and adding and subtracting, we find x and y.

There are many variants of this argument. For example, note that

$$x^5 + y^5 = (x^2 + y^2)(x^3 + y^3) - x^2y^2(x + y).$$

But

$$x^2 + y^2 = (x + y)^2 - 2xy \qquad \text{and} \qquad x^3 + y^3 = (x + y)^3 - 3xy(x + y).$$

Now let $p = xy$. Using the fact that $x + y = 1$ we find

$$11 = x^5 + y^5 = (1 - 2p)(1 - 3p) - p^2.$$

The above equation simplifies to $5p^2 - 5p - 10 = 0$, and again we find that $p = 2$ or $p = -1$.

Comment. Each method exploited symmetry. Often the only feasible way to solve a problem is to take advantage of symmetries. That is frequently true even in very applied problems. For example, the fact that gravitation is centrally symmetric is a key fact in deducing the motions of the planets. If there is symmetry, it is usually important to hold on to it as long as possible.

Problem 5. Call a set $\mathcal{S}$ of positive integers *multiple-rich* if for any positive integer n, some multiple of n (perhaps n itself) is in $\mathcal{S}$. For example, the set of positive perfect squares is multiple-rich, and the set of primes is not.

Suppose that the positive integers are divided into two teams, say $\mathcal{A}$ and $\mathcal{B}$. Show that at least one of $\mathcal{A}$ or $\mathcal{B}$ is multiple-rich.

Solution. Suppose that $\mathcal{A}$ is *not* multiple-rich. Then there is a positive integer r such that $\mathcal{A}$ does not contain any multiple of r.

Similarly, if $\mathcal{B}$ is not multiple-rich, there is a positive integer s such that $\mathcal{B}$ does not contain any multiple of s.

Let $t = rs$. Note that t is simultaneously a multiple of r and a multiple of s. So t cannot be in either $\mathcal{A}$ or $\mathcal{B}$. This is impossible, since $\mathcal{A}$ and $\mathcal{B}$ between them contain *all* the positive integers.

Thus the assumption that *neither* $\mathcal{A}$ nor $\mathcal{B}$ is multiple-rich leads us to something that cannot be true. It follows that this assumption is false, meaning that *at least* one of $\mathcal{A}$ or $\mathcal{B}$ is multiple-rich.

Comment. Note that more or less the same argument shows that if $\mathcal{S}$ is multiple-rich, and is divided into finitely many teams, then at least one of the teams is multiple-rich.

Was this an easy problem or a hard one? The proof was very short, much shorter than the other arguments in this problem set. But it can *feel* harder. There are no formulas to manipulate.

The argument is indirect, what is sometimes called a "proof by contradiction." We proved that the result holds by showing that assuming it does not hold leads to a conclusion that is demonstrably false, an absurdity, a contradiction.

"Proof by contradiction" is a basic tool in mathematics. The standard proofs of many famous results use this strategy. Two important examples are the proof that there are infinitely many primes, and the proof that $\sqrt{2}$ is not a rational number.

http://www.pims.math.ca/education/math_problems/
http://www.math.ubc.ca/~adler/problems/