Solutions to March 2006 Problems

Problem 1. Let $a_n = 1! + 2! + 3! + \cdots + n!$. For what positive integers n is a_n a perfect square? Hint: Calculate for a while.

Solution. We find a_n for the first few values of n: $a_1 = 1$, $a_2 = 3$, $a_3 = 9$, $a_4 = 33$, $a_5 = 153$, $a_6 = 873$, and $a_7 = 5913$. Note that a_1 and a_3 are perfect squares. We will show that a_n can not be a perfect square if $n \geq 4$.

Note that the last decimal digit of a_4, a_5, a_6, and a_7 is 3. We first show that the last digit of a_n is 3 for *any* $n \geq 4$. If $n \geq 5$, each term of

$$5! + 6! + \cdots + n!$$

is a multiple of 10. Thus if $n \geq 5$, then a_n is 33 plus a multiple of 10, and therefore the last digit of a_n is 3.

To finish, we show that *no* number with last digit 3 can be a perfect square. Let m be a non-negative integer. The last digit of m^2 is determined by the last digit of m. And if m has last digit 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, then m^2 has last digit 0, 1, 4, 9, 6, 5, 6, 9, 4, 1, thus never 3.

Another Way. In the first solution, we examined remainders when a_n is divided by 10. This is "natural" only because we have ten fingers, so use decimal notation, and the 3's at the end of a_4, a_5, a_6, and a_7 stand out.

A Martian might approach things differently. Recall that Martians have a single seven-fingered hand. It is easy to verify directly that a_n is not a perfect square if $n \leq 6$. If $n \geq 7$, then $7! + 8! + \cdots + n!$ is divisible by 7, and therefore

$$a_n = (1! + 2! + 3! + 4! + 5! + 6!) + 7M$$

for some integer M. But $1! + 2! + \cdots + 6!$ leaves a remainder of 5 on division by 7. Thus if $n \geq 7$, then $a_n = 7N + 5$ for some integer N.

However, $7N + 5$ can never be a perfect square. Note that $0^2 = 0$, $1^2 = 1$, $2^2 = 4$, 3^2 leaves a remainder of 2 on division by 7, 4^2 leaves a remainder of 2, 5^2 leaves a remainder of 4, and 6^2 leaves a remainder of 1. The remainder pattern 0, 1, 4, 2, 2, 4, 1 repeats forever, since in general the remainder when $(7k + a)^2$ is divided by 7 is the same as the remainder when a^2 is divided by 7. In particular, the remainder when a perfect square is divided by 7 is never 5. Since a_n has remainder 5 on division by 7 if $n \geq 7$, it can not be a perfect square.

This argument may seem harder than the first one. However, Martians use base 7 notation, and easily notice that a_n "ends" in a 5 for all $n \geq 6$, just like we noticed that the decimal expansion of a_n ends in a 3 for $n \geq 4$.

Comment. At the Interplanetary Number Theory Conference, a Venusian observed that if $n > 1$, then a_n can not be a perfect cube, or a perfect 4-th power, or a perfect 5-th power, and so on. Recall that Venusians have 3 hands, with 9 fingers on each hand. Thoughtfully, the Venusian translated her argument from base 27 to base 10 notation.

The Venusian began by noting that a_n is not a perfect k-th power for any $k \geq 3$ if $2 \leq n \leq 7$. Next she attacked the problem for $n \geq 8$. She observed that 9! is divisible by (Earth) 27, and therefore $m!$ is divisible by 27 for any $m \geq 9$. But $a_8 = 46233$, and therefore if $n \geq 8$,

$$a_n = 46233 + 27M$$

for some integer M. However, $46233 = 9 \times 5137$, and 5137 is not divisible by 3. Since $27M$ *is* divisible by 3^3, it follows that $46233 + 27M$ is divisible by 3^2 but by no higher power of 3. In particular, it can not be a perfect k-th power for any $k \geq 3$.

Problem 2. Alphonse and Beth ran one lap on the school 400 meter track. They started at the same time and place but ran in opposite directions, each at constant speed. Beth finished her lap 27.2 seconds after they passed each other, and Alphonse finished 42.5 seconds after they passed. How long did Beth take to run the lap? (A solution with some symmetry would be nice. There is even a geometric solution.)

Solution. Let t be Beth's lap time. Then Alphonse's time is $t + (42.5 - 27.2)$. Their running speeds, in meters per second, are therefore

$$\frac{400}{t} \qquad \text{and} \qquad \frac{400}{t + 15.3}.$$

The distances they ran after they passed are

$$(27.2)\frac{400}{t} \qquad \text{and} \qquad (42.5)\frac{400}{t + 15.3}.$$

These distances add up to 400. After simplifying the resulting equation, we obtain $t^2 - 54.4t - 416.16 = 0$. By the Quadratic Formula, $t = 61.2$.

Another Way. More generally, suppose Alphonse finishes a seconds after they pass, and Beth finishes b seconds after they pass. Let p be Alphonse's speed, let q be Beth's, and let t be the time they run *before* they meet. So when they meet Alphonse has run a distance pt, and Beth has run a distance qt. To finish, Alphonse needs to run distance qt, which takes time qt/p. Similarly, Beth needs time pt/q to finish. We were told that

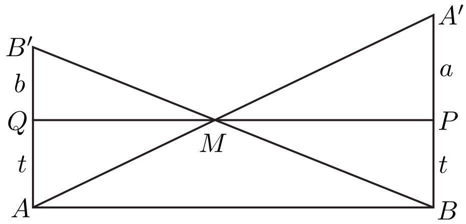
$$\frac{qt}{p} = a \qquad \text{and} \qquad \frac{pt}{q} = b.$$

Multiply. We get $t^2 = ab$, so $t = \sqrt{ab}$. Thus Beth's running time is $\sqrt{ab} + b$. Finally, put $a = 42.5$ and $b = 27.2$. Beth ran the lap in 61.2 seconds.

Comment. Nice run! The first solution is efficient, but we prefer the symmetry of the second. Working with "general" a and b is more informative (and simpler!) than working with particular numbers. That happens often. Note that the track length was irrelevant.

Another Way. This is a geometric version of the previous solution. The track is oval, but it is easier to represent it as a line segment. So cut the track at the start point and straighten it out.

In the picture below, the horizontal direction represents position, and the vertical direction represents time. A point in the plane of the picture has two coordinates, a position or “space” coordinate, and a time coordinate, so it is a point in space-time.

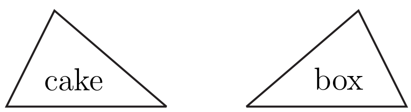


Let Alphonse start at space-time point A. Beth starts at the same place and time as Alphonse, but because she is running in the other direction, think of her start position as being at the other end of the straightened out track. Her start time is the same as Alphonse's. So she starts at space-time point B, where $AB = 400$.

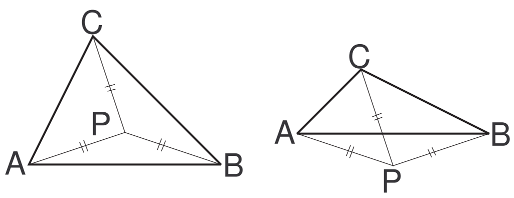
The line segment AA' represents Alphonse's path in space-time, while BB' represents Beth's. (The paths are line segments because speeds are constant.) Note that M is the point in space-time where they pass. Let the time when they pass be t, let the amount of time that Alphonse runs after they pass be a, and let the amount of time Beth runs after they pass be b.

Note that $\triangle BA'M$ is a scaled up version of $\triangle B'AM$, and the same scaling factor produces $\triangle PA'M$ from $\triangle QAM$, and also $\triangle PBM$ from $\triangle QB'M$. It follows that $a/t = t/b$, and therefore $t = \sqrt{ab}$. So from the picture we see that Alphonse's running time is $\sqrt{ab} + a$ while Beth's is $b + \sqrt{ab}$.

Problem 3. A mathematician moonlighting as a pastry chef has made a triangular cake, frosted on top but not on the sides. The box for the cake fits perfectly, but only if the cake is put in upside down. Show how to cut the cake into some pieces so that the cake can be put in the box right side up. Try to find more than one solution.



Solution. The first method is illustrated in the figure below. Draw the perpendicular bisector of side AB of the triangle (this bisector is not shown). The points on the bisector are the points *equidistant* from A and B.



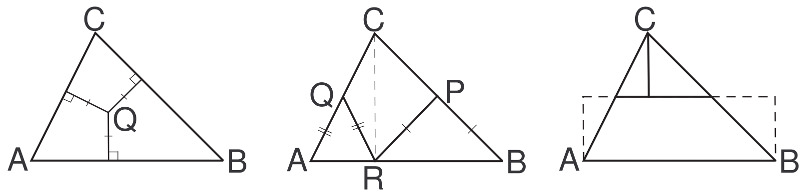
Draw the perpendicular bisector of side BC. The points on this bisector are the points equidistant from B and C. Let the perpendicular bisectors of AB and of BC meet at P. Then P is equidistant from A, B, and C. In particular, P is on the perpendicular bisector of side CA. We have proved the interesting fact that the perpendicular bisectors of the three sides meet at one point.

The point P is called the *circumcenter* of $\triangle ABC$. Since P is equidistant from A, B, and C, the circle with center P and radius AB is the circle *circumscribed* about $\triangle ABC$. Note that if $\triangle ABC$ is acute (left-hand diagram above), then P lies in $\triangle ABC$, but P lies outside $\triangle ABC$ if the triangle is obtuse (right-hand diagram).

For now concentrate on the left-hand diagram, which represents the acute case. Cut the cake along line segments PA, PB, and PC. This divides the cake into 3 pieces. The top of each piece is an isosceles triangle. Temporarily slide the piece PCA out of the way. Then rotate the piece PBC counterclockwise about P until C ends up in position A. Finally, slide the old piece PCA into the "hole" on the right side of the cake. We have transformed the original cake into its mirror image without turning anything upside down and spoiling the frosting. Now the cake fits in the box perfectly.

The same procedure works in the obtuse and the right-angled case (left-hand diagram). Here the situation is in a sense simpler, since we make only one cut, but is a little harder to visualize. The triangle PBA can be thought of as a piece of "virtual cake" that we adjoin during the transformation process, and then remove.

Another Way. The next method is illustrated in the left-hand part of the diagram below. Imagine drawing the bisector of $\angle CAB$ (this bisector is not shown). The points on the angle bisector are the points equidistant from the lines AB and AC.



Draw the bisector of $\angle ABC$. Any point on this bisector is equidistant from the lines BA and BC. Let the two angle bisectors meet at Q. Then Q is equidistant from AB, BC, and CA. Since Q is equidistant from CA and CB, it lies on the bisector of $\angle BCA$. We have shown that the angle bisectors of a triangle meet at a point.

The (perpendicular) distances from Q to the three sides are all the same, say r. It follows that the circle with center Q and radius r is the inscribed circle of $\triangle ABC$. For this reason, Q is called the *incenter* of $\triangle ABC$.

Draw the three line segments through Q perpendicular to the three sides of $\triangle ABC$, as in the diagram, and cut the cake along these lines. Our cuts divide the triangle into three symmetrical "kites." It is easy to rearrange these kites into the mirror image of the original triangle.

Another Way. For the next method, look at the middle diagram. Here the triangle has been placed with a longest edge at the bottom. Drop a perpendicular (the dashed line) from the top vertex to the bottom edge and let it meet the bottom at R. Let P be the point on BC such that $RP = BP$, and let Q be the point on AC such that $RQ = AQ$. Draw the line segments RP and RQ.

A bit of "angle-chasing" shows that P bisects BC and Q bisects AC. First we deal with P. Let $\angle PRB = \angle PBR = x$. Then $\angle CRP = 90 - x$. But $\angle PCR + \angle PBR = 90$, so $\angle PCR = 90 - x$. It follows that $PC = PR$, and therefore $PC = PB$. The argument for Q is the same.

Cut the cake along the lines RP and RQ. We get three pieces, two of them isosceles triangles and the third a "kite." Each piece has mirror symmetry. It is easy to rearrange these pieces (without turning over) to make the mirror image of $\triangle ABC$.

Another Way. The last method we give is not based on cutting into pieces with mirror symmetry. Look at the right-hand picture in the figure above. The triangle has been placed so that so that neither bottom angle is obtuse.

Draw a line parallel to the bottom and halfway up the triangle, and drop a perpendicular from the top vertex to that line. Cut the triangle along these lines. The diagram shows how to rearrange the pieces to make a rectangle: rotate the top two pieces by $180°$ and slide them into position.

If we had started with the *mirror image* of the original triangle and used the same cutting process, we would arrive at *the same rectangle.* Reverse this last process: the rectangle can be cut into pieces and reassembled to make the mirror image of $\triangle ABC$. So the full process goes like this. Cut $\triangle ABC$ and reassemble the pieces to form a rectangle $\mathcal{R}$. Now cut $\mathcal{R}$ into pieces that can be reassembled to make the mirror image of $\triangle ABC$. Note that the cuts used to make $\mathcal{R}$ from $\triangle ABC$ are still there, so the pieces that were used get further subdivided.

Comment. The first three subdivisions were geometrically simple—they may be the only really simple methods. But there are infinitely many ways to cut $\triangle ABC$ into polygonal pieces and reassemble them to make the mirror image.

There is an interesting related result called *Bolyai-Gerwien Theorem.* Let $\mathcal{A}$ and $\mathcal{B}$ be *any* two polygons with the same area. Then $\mathcal{A}$ can be cut up into a finite number of polygonal pieces that can be reassembled, without turning any piece over, to make $\mathcal{B}$. So for example if the cake box has a square top with the same top area as the cake, then the triangular cake can be cut into pieces that fit this new box perfectly.

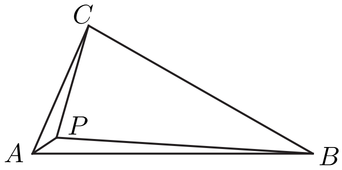
Problem 4. (a) Show that if $\triangle ABC$ is equilateral, then for any point P in the interior of $\triangle ABC$, the line segments PA, PB, and PC can be rearranged to form a triangle. (b) Show that if $\triangle ABC$ is not equilateral, there is a point P in the interior of the triangle such that the line segments PA, PB, and PC can not be rearranged to form a triangle.

Solution. (a) Recall that three positive numbers are the sides of a triangle if and only if the sum of any two is greater than the third. Without loss of generality we may assume that the sides of $\triangle ABC$ are equal to 1. In what follows, $|PQ|$ denotes the length of the line segment PQ.

Since the points P, A, and B are the vertices of a triangle, we have $|PA| + |PB| > |AB| = 1$. Since P is an interior point of $\triangle ABC$, we have $|PC| < 1$. Thus $|PA| + |PB| > |PC|$.

Similarly, $|PB| + |PC| > |PA|$ and $|PC| + |PA| > |PB|$. So the sum of any two of $|PA|$, $|PB|$, and $|PC|$ is bigger than the third. It follows that the segments PA, PB, and PC can be arranged to form a triangle.

(b) If necessary, relabel the vertices so that $|AB| \geq |BC| \geq |CA|$. Since $\triangle ABC$ is not equilateral, $|AB| > |CA|$. Choose a point P in the interior of $\triangle ABC$ which is "very close" to A. More informally, let P be a point very close to the vertex where a longest and a shortest side meet. (There may be two longest sides, or two shortest sides.) The situation is illustrated in the picture below.



It is clear from the picture that $|PA|+|PC| < |PB|$, which means that the line segments PA, PB, and PC can not be rearranged to form a triangle.

But what about other possible pictures? Suppose that AC is a shortest side, AB is a longest side, and the triangle is not equilateral, which means that $|AB| > |AC|$. Is it really obvious that $|PA|+|PC| < |PB|$ as long as P is "close enough" to A?

One way to think about it is to imagine an ant crawling inside the triangle towards A. Let X represent the position of the moving ant. Suppose there are points $X \neq A$ but arbitrarily close to A with $|XA| + |XC| \geq |XB|$. Then we must have $|AA|+|AC| \geq |AB|$, which is false, since $|AA| = 0$ and $|AC| < |AB|$.

Maybe the above argument is *too* informal—and we should not need ants to do mathematics. We give a more precise argument that avoids vague terms such as "close enough." Let ϵ be a positive number, which we should think of as small (we will soon see how small), and let P be a point in the interior of our triangle with $|PA| = \epsilon$. Since P, A, and B form a triangle, we have $|PB| + |PA| > |AB|$, so

$$|PB| > |AB| - \epsilon. \tag{1}$$

Since P, A, and C form a triangle, we have $|AC| + |PA| > |PC|$, so

$$|PC| < |AC| + \epsilon. \tag{2}$$

We want to show that an ϵ can be found such that $|PA| + |PC| < |PB|$. From Inequality 2, we obtain

$$|PA| + |PC| < |AC| + 2\epsilon \tag{3}$$

(informally, $|PA| + |PC|$ can not be much bigger than $|AC|$.)

But from Inequality 1, $|PB| > |AB| - \epsilon$ ($|PB|$ can not be much smaller than $|AB|$), and therefore we will have $|PA| + |PC| < |PB|$ if

$$|AC| + 2\epsilon < |AB| - \epsilon,$$

or equivalently if $\epsilon < (|AB| - |AC|)/3$. Since AC is a shortest side and AB is a longest side, and the triangle is not equilateral, we have $|AB| > |AC|$, so $|AB| - |AC|$ is positive, and there are (many) positive ϵ less than $(|AB| - |AC|)/3$. The vague phrase "close enough" has been made precise: less than $(|AB| - |AC|)/3$ is definitely close enough.

We have shown that if $|PA| < (|AB| - |AC|)/3$, then $|PA| + |PC| < |PB|$, and so the line segments PA, PB, and PC can not be rearranged to form a triangle. This completes the proof.

Comment. The above argument may seem hard. If you look at it closely, you will find that it is not so hard. The argument could be made much shorter: it is long partly because we tried to describe the motivation for each step. In school, we get a lot of experience with equations, but much less with inequalities. So arguments that make heavy use of inequalities can look harder than they are.

Problem 5. In how many ways can we express 90000 as a sum of consecutive odd integers?

Solution. If 90,000 is expressed as the sum of odd numbers, consecutive or not, we must use an *even* number of odd numbers, since the sum of an odd number of odd numbers is odd. Suppose we use $2n$ odd numbers. Arrange the numbers in increasing order, and let the first one be $2a + 1$. Then the second (if there is one) is $2a + 3$, the third is $2a + 5$, and so on. Finally, the $2n$-th number is $2a + 1 + 2(2n - 1)$.

Add together the first number and the last, the second and the next to last, and so on, until we add the n-th and the $(n + 1)$-th. These sums of pairs are all equal to $4a + 4n$. Thus the sum of all the numbers is $(4a + 4n)(n)$, and we arrive at the equation

$$4n(a + n) = 90000, \qquad \text{or equivalently} \qquad n(a + n) = 22500.$$

We look for integer solutions of this equation with n positive. (Note that a need not be positive.)

We will find the number of positive integer solutions of $xy = 22500$. Given any such solution, if we put $n = x$ and $a = y - x$, we obtain a representation of 90000 as a sum of consecutive odd numbers. Conversely, any representation of 90000 as a sum of consecutive odd numbers yields a solution of $xy = 22500$.

Listing the solutions and then counting is not difficult. But it is not pleasant, for there are 45. We find an easier way of counting. Express 22500 as a product of primes. We get

$$22500 = 2^2 3^2 5^4.$$

The divisors of 22500 are the numbers of the form $2^a 3^b 5^c$, where $0 \le a \le 2$, $0 \le b \le 2$, and $0 \le c \le 4$. To make a divisor of 22500, first look at the prime 2, and decide how many 2's the divisor will "get:" 0, 1, or 2. There are 3 ways to make the decision. For *every* such way, there are 3 ways to decide how many 3's the divisor will get. So there are (3)(3) ways to decide how many 2's and how many 3's the divisor will get. For every decision about the 2's and 3's, there are 5 decisions for how many 5's the divisor will get. So 22500 has (3)(3)(5) positive divisors, giving 45 choices for x. Once x is chosen, there is only one y that works. Thus there are 45 ways to express 90000 as a sum of consecutive odd numbers.

Comment. We can use the same basic idea, with occasional mild complications, to find the number of representations of any N as a sum of consecutive odd numbers, or as a sum of consecutive numbers, or as a sum of consecutive numbers of the form $3k + 1$, and so on.

We might ask instead for the number of ways to express 90000 as a sum of consecutive *positive* odd numbers. First note that $22500 = 150^2$. This gives, in the notation we used above, $n = 150$ and $a = 0$. So 90000 can be expressed as the sum of the *first* 300 odd numbers.

Now look at any *other* representation of 90000 as a sum of consecutive positive odd numbers, with the smallest number being $2b + 1$, where $b \neq 0$. Throw in all the positive odd numbers less than $2b + 1$, *together with* all their negatives. The numbers thrown in add up to 0, so we get a representation of 90000 as a sum of consecutive odd numbers, some of which are negative. It is easy to see that in this way we get all representations of 90000 as a sum of consecutive odd numbers, some of which are negative.

Thus there are 44/2 representations of 90000 as a sum of consecutive odd numbers all greater than 1, and therefore 23 representations as a sum of consecutive positive odd numbers.

More simply, we want to count the number of pairs (x, y) of positive integers such that $xy = 22500$ and $x \leq y$. The pair $(150, 150)$ works. Of the remaining 44 pairs of positive integers such that $xy = 22500$, half have $x < y$ and half have $x > y$. So we get a total of $1 + 44/2$ representations.