Solutions to February 2008 Problems

Problem 1. An Egyptian-style pyramid has a square $b \times b$ base and height h. What is the volume of the smallest sphere that contains the pyramid? (This may be a little trickier than it looks.)

Solution. Let P be the apex of the pyramid, and let M be the midpoint of the base. Let A be any one of the 4 corners of the base. *If* we can find a point O on the line segment PM which is equidistant from P and A, then it is geometrically reasonably clear that the sphere with center O and radius OP is the smallest sphere that contains the pyramid. And it is also reasonably clear that if h is fairly "large" compared to b, then there will be such a point O.

We now look for such a point O, with $r = OP$. Then $OM = h - r$. We want to make sure that the distance from O to any corner of the base (say A) is r.

Since $\triangle OMA$ is right-angled at M, by the Pythagorean Theorem we must have $(OM)^2 + (AM)^2 = (OA)^2$. But by an easy application of the Pythagorean Theorem, or otherwise, we can see that $(AM)^2 = b^2/2$. Thus we need to have

$$(h - r)^2 + b^2/2 = r^2.$$

Expand the first term, and solve for r. We obtain

$$r = \frac{h^2 + \frac{b^2}{2}}{2h} = \frac{h}{2} + \frac{b^2}{4h}. \tag{1}$$

This value of r works, *provided that* the intuitive geometric idea we have used is right, that is, provided that the point O really is on the line segment PM. This will be the case as long as $r \le h$. Using Equation 1, we conclude that we have found the appropriate r, *unless* $h/2 + b^2/4h > h$, or equivalently unless $b^2 > 2h^2$.

Note that $b^2 = 2h^2$ precisely if $h = AM$. Any sphere that contains the pyramid must in particular contain the 4 vertices of the base. So any such sphere must have radius at least equal to AM. If $h < AM$, then the sphere with center M and radius AM will contain the pyramid, and no smaller sphere will work. So finally we conclude that the desired sphere has radius r given by Equation 1 if $h \ge b/\sqrt{2}$, and radius $r = b/\sqrt{2}$ otherwise. And now that we have found the suitable radius, the volume is easy to find.

Problem 2. For any positive integer n, let $f(n)$ be the integer nearest to $\sqrt{n}$. Evaluate

$$\frac{1}{f(1)} + \frac{1}{f(2)} + \frac{1}{f(3)} + \frac{1}{f(4)} + \frac{1}{f(5)} + \cdots + \frac{1}{f(9999)} + \frac{1}{f(10000)}.$$

Solution. We could do a long computation, or else write a program for a computer or a programmable calculator to do the computation for us. But presumably that is not what is being asked for, there must be a better way.

It may be useful to calculate $f(1)$, $f(2)$, $f(3)$, and so on for a while in order to see whether the problem is easier than it looks. Easily, $f(1) = f(2) = 1$. Also, $f(3) = f(4) \cdots = f(6) = 2$, $f(7) = f(8) = \cdots = f(12) = 3$, $f(13) = f(14) = \cdots = f(20) = 4$, $f(21) = f(22) = \cdots = f(30) = 5$, $f(31) = f(32) = \cdots = f(42) = 6$, and $f(43) = f(44) = \cdots = f(56) = 7$, while $f(57) = 8$.

Interesting! There are 2 numbers n such that $f(n) = 1$; there are 4 numbers n such that $f(n) = 2$; there are 6 numbers n such that $f(n) = 3$; there are 8 such that $f(n) = 4$; there are 10 such that $f(n) = 5$, and 12 such that $f(n) = 6$, and 14 such that $f(n) = 7$. The pattern (so far) is clear. And we can easily see, for example, that

$$\frac{1}{f(1)} + \frac{1}{f(2)} + \frac{1}{f(3)} + \frac{1}{f(4)} + \frac{1}{f(5)} + \cdots + \frac{1}{f(55)} + \frac{1}{f(56)} = (2)(7).$$

We need to verify that the pattern we have observed indeed continues. Obviously $f(n) = a$ when n is "close" to a^2. Now we should decide exactly what "close" means.

We have $f(n) = a$ precisely when $(a - 1/2)^2 < n < (a + 1/2)^2$. (The square root of n can never be $k + 1/2$, where k is an integer, since $(k + 1/2)^2 = k^2 + k + 1/4$, and $k^2 + k + 1/4$ cannot be an integer.)

But $(a - 1/2)^2 < n < (a + 1/2)^2$ if and only if

$$a^2 - a + 1/4 < n < a^2 + a + 1/4.$$

Since n must be an integer, the above inequalities are equivalent to

$$a^2 - a + 1 \leq n \leq a^2 + a.$$

The number of integers in the interval from $a^2 - a + 1$ to $a^2 + a$ (inclusive) is $2a$, so the pattern we observed does indeed continue.

Now we can calculate. We have 2 values of n for which $f(n) = 1$, 4 values for which $f(n) = 2$, and so on, up to (2)(99) values for which $f(n) = 99$. Our sum so far is (2)(99). And finally, $f(n) = 100$ for n (in our range) going from $100^2 - 100 + 1$ to 100^2, a total of 100 values. This makes a contribution of 1 to our sum. Thus our sum is 199.

Problem 3. Show that the equation $x^3 + 2y^3 + 4z^3 = 0$ has exactly one solution in integers.

Solution. We have the obvious solution $x = y = z = 0$. We need to show that there are no others. The idea will be quite simple, but we must be careful in writing it up.

Suppose that x, y, and z are integers such that $x^3 + 2y^3 + 4z^3 = 0$. Since $x^3 = -(2y^3 + 4z^3)$, it follows that x^3 is even. Thus x must be even. Let $x = 2x_1$. We conclude that $8x_1^3 + 2y^3 + 4z^3 = 0$, and therefore $y^3 + 2z^3 + 4x_1^3 = 0$.

The argument used above now shows that y is even, say $y = 2y_1$. After a little while we find that $z^3 + 2x_1^3 + 4y_1^3 = 0$.

Thus z is even. Let $z = 2z_1$. Then $x_1^3 + 2y_1^3 + 4z_1^3 = 0$. Now repeat the argument, using x_1, y_1, and z_1. We find that x_1, y_1, and z_1 are even. Let $x_1 = 2x_2$, $y_1 = 2y_2$, and $z_1 = 2z_2$. Then $x_2^3 + 2y_2^3 + 4z_2^3 = 0$.

But then x_2, y_2, and z_2 are even. Let $x_2 = 2x_3$, $y_2 = 2y_3$, and $z_2 = 2z_3$. Then $x_3^3 + 2y_3^2 + 4y_3^3 = 0$.

Continue. We saw that $x = 2x_1$ for some integer x_1, and $x_1 = 2x_2$ for some integer x_2, and $x_2 = 2x_3$ for some integer x_3, and so on forever. It follows that x is divisible by 2, by 4, by 8, and so on forever. But the only integer that is divisible by every power of 2 is 0. So $x = 0$. The same argument shows that $y = 0$ and $z = 0$. This completes the proof.

Another Way. The same basic idea can be written up in a more efficient way. Suppose that $x^3 + 2y^3 + 4z^3 = 0$, where the numbers x, y, and z are not all equal to 0. By dividing x, y, and z by the same appropriate power of 2, we can find integers u, v, and w such that $u^3 + 2v^3 + 4w^3 = 0$ and at least one of u, v, and w is odd. However, the argument of the first solution shows that u, v, and w must all be even. Thus there cannot be a solution (x, y, z) other than $(0, 0, 0)$.

Problem 4. Let S be a set of 51 integers chosen from $\{0, 1, 2, \ldots, 99\}$. Show that there are two integers in S which differ by exactly 10.

Solution. We will show that if no two integers in S differ by 10, then $\mathcal{S}$ must have 50 or fewer elements.

Let A consist of all numbers in S that lie between 0 and 9, or between 20 and 29, or between 40 and 49, or between 60 and 69, or between 80 and 89—the even decades. Let B consist of all numbers in S that lie between 10 and 19, or between 30 and 39, or between 50 and 59, or between 70 and 79, or between 90 and 99—the odd decades.

Suppose that no two integers in S differ by 10. Then for any element a of A, the number $a + 10$ cannot be in B. So if A has n elements, then B can have at most $50 - n$ elements, for a total of at most 50.

Another Way. We give essentially the same proof, using somewhat different language. Let S_0 be the collection of numbers in S which lie between 0 and 19 (inclusive). Let S_1 be the collection of numbers in S which lie between 20 and 39, and let S_2 be the collection of numbers in S which lie between 40 and 59. Define S_3 and S_4 analogously.

We have divided S into 5 sets S_i, which between them have 51 elements. It follows that at least one of the S_i has 11 or more elements. Assume for concreteness (it doesn't matter) that there are 11 or more elements of S between 20 and 39. The last digit of the decimal representation of a number must be one of 10 digits 0, 1, 2, ..., 9. So if S has 11 or more elements between 20 and 39, there must be 2 of them that end with the same digit, meaning that they differ by 10.

Comment 1. Consider the 50 integers 0, 1, ..., 9, 20, ..., 29, 40, ..., 49, 60, ..., 69, 80, ..., 89. It is easy to see that no two of these differ by exactly 10. Thus 51 cannot be replaced by any smaller number.

Problem 5. A fair die is tossed n times. Find a *simple* expression for the probability that the number of 6's obtained is even.

Solution. Record the result of the tossing as a "word" of length n, made up of "letters" chosen from $\{1, 2, 3, 4, 5, 6\}$. All words are equally likely, and there are 6^n of them. We will count how many of these words have an even number of 6's.

Let $E(n)$ be the number of words with an even number of 6's. For the sake of symmetry, it is useful to let $O(n)$ be the number of words with an odd number of 6's. It is certainly not necessary to introduce $O(n)$, since clearly $O(n) = 6^n - E(n)$, but it makes life more pleasant.

It is clear that $E(1) = 5$, since there are 5 1-letter words that have an even number of 6's: the words 1, 2, 3, 4, and 5 all have an even number of 6's. It is also clear that $O(1) = 1$. We could, and probably should, compute $E(2)$, $O(2)$, $E(3)$, and so on for a little while, to get some feeling for the situation.

For example, the words of length 2 that have an even number of 6's are the words with 0 6's (there are 5^2 of them), together with the word 66. So $E(2) = 26$, and therefore $O(2) = 10$. Similarly, the words of length 3 that have an even number of 6's are the words with no 6's (there are 5^3 of them), together with the words with 2 6's and any non-6. To count these, note that the non-6 can be chosen in 5 ways, and for every such choice, *where* it goes can be chosen in 3 ways. Now we are forced to fill the remaining 2 spots with 6's. Thus there are 15 words with exactly 2 6's. It follows that $E(3) = 140$, and therefore $O(3) = 76$.

Now we show how to compute in a more streamlined way. Suppose that we know $E(n)$ and $O(n)$. Can we find a simple expression for $E(n + 1)$ and $O(n + 1)$?

A word of length $n+1$ with an even number of 6's can arise in the following two ways: (i) We put a non-6 at the end of a word of length n that has an even number of 6's, or (ii) We put a 6 at the end of a word of length n that has an odd number of 6's.

The number of words of type (1) is $5E(n)$, while the number of words of type (ii) is $O(n)$. We have just shown that $E(n+1) = 5E(n)+O(n)$. A similar argument shows that $O(n + 1) = E(n) + 5O(n)$. We line up these results to admire them.

$$E(n + 1) = 5E(n) + O(n) \tag{2}$$
$$O(n + 1) = E(n) + 5O(n) \tag{3}$$

The first immediate benefit of these equations is that we can now compute, in a relatively trouble-free way, $E(n)$ and $O(n)$ for n that are not too large. A second benefit, for someone acquainted with matrices, is that there is an easy way to write the two equations as a single equation involving a matrix. We will not explore that, though in other more complicated situations the matrix approach can be very valuable.

Instead, "subtract" the second equation from the first. We get

$$E(n+1) - O(n+1) = 4(E(n) - O(n)).$$

If we let $D(k)$ (for *difference*) be $E(k) - O(k)$, we see that $D(n+1) = 4D(n)$. As we increment n by 1, the difference gets multiplied by 4. From our earlier calculation, $E(1) - O(1) = 4$. It follows that in general $E(n) - O(n) = 4^n$.

(Actually, the result also applies at $n = 0$. For the *empty word* has 0 6's, an even number, so $E(0) = 1$. And it is easy to see that $O(0) = 0$. Thus $E(0) - O(0) = 1 = 4^0$. Who cares? I kind of do.)

We know that $E(n) - O(n) = 4^n$, and of course we always knew that $E(n) + O(n) = 6^n$. Add, divide by 2. We find that $E(n) = (6^n + 4^n)/2$.

Finally, to find the desired probability, divide by 6^n. After we simplify a little, we get the expression

$$\frac{1}{2}\left(1 + \left(\frac{4}{6}\right)^n\right).$$

Note that our probability approaches 1/2 quite quickly, which is in accord with intuition. If n is large, an even number of 6's and an odd number of 6's should be roughly equally likely. The probability of an even number of 6's is, however, always greater than the probability of an odd number of 6's.

Comment 2. One can give essentially the same argument directly with probabilities. Take an experiment which has probability of "success" p, and therefore probability of failure q, where $q = 1 - p$. Suppose that we repeat the experiment independently n times. We want to calculate $e(n)$, the probability that the number of successes is even.

Let $o(n)$ be the probability that the number of successes is odd. We find an expression for $e(n+1)$ in terms of $e(n)$ and $o(n)$. The number of successes after $n+1$ trials is even if (i) we have an even number of successes after n trials, and then a failure, or if (ii) we have an odd number of successes after n trials and then a success. The probability of (i) is $qe(n$, and the probability of (ii) is $po(n)$. It follows that $e(n+1) = qe(n) + po(n)$. Similarly, $o(n+1) = pe(n) + qo(n)$.

Note that the argument here is more abstract than the counting argument of the first solution. But there is greater generality. Subtract as in the first solution. We find that $e(n+1) - o(n+1) = (q-p)(e(n) - o(n))$. From the fact that $e(1) = p$ and $o(1) = q$ we conclude that $e(n) - o(n) = (q-p)^n$. Then since $e(n) - o(n) = 1$, we find that $e(n) = (1/2)(1 + (q-p)^n)$. Our problem is the case $p = 1/6$, $q = 5/6$.

Another Way. Like in the first solution, we want to find the number of "words" of length n over the alphabet $\{1, 2, 3, 4, 5, 6\}$ that have an even number of 6's.

We find a general expression for the number words that have exactly k 6's. The location of these k 6's can be chosen in $\binom{n}{k}$ ways, and for each such choice, the remaining $n - k$ positions can be filled with numbers chosen from $\{1, 2, 3,$

4, 5} in 5^{n-k} ways, so there are $\binom{n}{k}5^{n-k}$ ways to get k 6's. It follows that the number of ways to get an even number of 6's is

$$\binom{n}{0}5^n + \binom{n}{2}5^{n-2} + \binom{n}{4}5^{n-4} + \cdots . \tag{4}$$

The expression above is not precisely simple, and if n is large, it does not lend itself to easy computation. However, we get some help from the Binomial Theorem. Note that

$$(5+1)^n = \binom{n}{0}5^n + \binom{n}{1}5^{n-1} + \binom{n}{2}5^{n-2} + \binom{n}{3}5^{n-3} + \binom{n}{4}5^{n-4} + \cdots ,$$
$$(5-1)^n = \binom{n}{0}5^n - \binom{n}{1}5^{n-1} + \binom{n}{2}5^{n-2} - \binom{n}{3}5^{n-3} + \binom{n}{4}5^{n-4} - \cdots .$$

Add, divide by 2. We find that Expression 4 is equal to $(6^n + 4^n)/2$. Now the probability is calculated like before.

Comment 3. Again, we could have worked directly with probabilities, putting $(5/6)^{n-k}$ for every occurrence of 6^{n-k} in Expression 4. We end up looking at the Binomial Theorem expansions of $(\frac{5}{6} + \frac{1}{6})^n$ and $(\frac{5}{6} - \frac{1}{6})^n$, and conclude that the required probability is $(1 + (4/6)^n)/2$.