Solutions to February 2007 Problems

Problem 1. Let $M = (1/2, 1/2)$. Find points A and B on the curve $y = 1/x^2$ such that M is the midpoint of the segment AB.

Solution. Make an informal sketch of the curve. A little play shows that there are such points A and B, and that they must lie on opposite sides of the y-axis. This geometric information will serve as a useful check on the subsequent algebra.

Let the x-coordinates of A and B be u and v. The y-coordinates are then $1/u^2$ and $1/v^2$. From the usual expression for the point midway between two given points, we get $(u + v)/2 = 1/2$ and $(1/u^2 + 1/v^2)/2 = 1/2$. So we want

$$u + v = 1 \qquad \text{and} \qquad \frac{1}{u^2} + \frac{1}{v^2} = 1.$$

The system takes some effort to solve. One way is to rewrite the second equation as $u^2 + v^2 = (uv)^2$. By squaring both sides of $u + v = 1$, we find that $u^2 + v^2 = 1 - 2uv$. Thus $(uv)^2 = 1 - 2uv$, and by the quadratic formula $uv = -1 \pm \sqrt{2}$.

Things are now routine. From the identity $(u - v)^2 = (u + v)^2 - 4uv$, we conclude that $(u - v)^2 = 5 + 4\sqrt{2}$ (one solution was rejected because $(u - v)^2$ can't be negative).

So $u - v = \pm\sqrt{5 + 4\sqrt{2}}$. Now that we know $u + v$ and $u - v$, u and v are easy to find. Since there is a solution, by symmetry there are 2 possible values of u, so both our solutions must be valid.

Another Way. We can preserve symmetry yet get down to one variable by letting $u = 1/2 - t$ and $v = 1/2 + t$. The equation $1/u^2 + 1/v^2 = 1$ becomes

$$\frac{1}{(1/2 - t)^2} + \frac{1}{(1/2 + t)^2} = \frac{1/2 + 2t^2}{(1/4 - t^2)^2} = 1.$$

Simplify. We obtain a quadratic equation in t^2, and now we can find t in the usual way.

Problem 2. Simplify $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + 4 \cdot 5 + \cdots + 99 \cdot 100$. (We could find the answer by doing a long calculation, or by writing a program to do it. That's not what is wanted here.)

Solution. It may be worthwhile to experiment, to add up terms of the series until we notice a pattern, if there is a pleasant one. A bit of notation is useful: let S_n be the sum if we add all the way to the term $n(n + 1)$. We want S_{99}.

Calculate. We get $S_1 = 2$, $S_2 = 8$, $S_3 = 20$, $S_4 = 40$, $S_5 = 70$. Is there anything obvious? Maybe not. Or else we might notice that $S_1 = (1 \cdot 2 \cdot 3)/3$, $S_2 = (2 \cdot 3 \cdot 4)/3$, $S_3 = (3 \cdot 4 \cdot 5)/3$, and so on at least for a little while. Even if we do notice this, we will need to verify that the pattern continues, that is, that $S_n = (1/3)(n)(n+1)(n+2)$ for all n. This could be done by induction, but there is a better way of seeing what is going on.

Let $a_k = k(k+1)(k+2)/3$. Note that

$$a_k - a_{k-1} = \frac{k(k+1)(k+2)}{3} - \frac{(k-1)(k)(k+1)}{3} = k(k+1),$$

and therefore

$$S_n = (a_1 - a_0) + (a_2 - a_1) + (a_3 - a_2) + \cdots + (a_n - a_{n-1}).$$

Almost everything cancels, and $a_0 = 0$, so $S_n = a_n$.

Comment. The "cancellation" trick we just used *always* works (when we think we know the formula). Do try it with your favourite sum, it is magical. For some reason I do not understand, collapsing sums are presented as a special purpose trick only.

Another Way. The "pattern" we found is admittedly a little hard to spot, so let's explore other ways of doing the problem.

Note that $(k+1)^3 - k^3 = 3k^2 + 3k + 1$, and therefore $k(k+1) = (1/3)[(k+1)^3 - k^3 - 1]$.

Sum $(k+1)^3 - k^3$ from $k = 1$ to $k = n$. We get

$$(2^3 - 1^3) + (3^3 - 2^3) + (4^3 - 3^3) + \cdots + ((n+1)^3 - n^3).$$

Lots of cancellation! We get $(n+1)^3 - 1$. So

$$S_n = \frac{1}{3}((n+1)^3 - 1 - n).$$

We can expand and "simplify" to $(1/3)(n^3+3n^2+2n)$, or $(1/3)(n)(n+1)(n+2)$.

Another Way. The identity $(k+1)^3 - k^3 = 3k^2 + 3k + 1$ was kind of pulled out of the air. We will arrive at a formula for S_n in another way. Since we are summing things of the form $k(k+1)$, which are quadratics, let's guess that S_n is a cubic in n, say

$$S_n = an^3 + bn^2 + cn + d.$$

Start with $n = 0$. Since clearly $S_0 = 0$, we conclude that $d = 0$. Since $S_1 = 2$, we have $2 = a + b + c$. Similarly, since $S_2 = 8$, we have $8 = 8a + 4b + 2c$. And since $S_3 = 20$, we have $20 = 27a + 9b + 3c$. This gives us three linear equations in the unknowns a, b, and c. These equations are not hard to solve: we get $a = 1/3$, $b = 1$, and $c = 2/3$. We arrive at the conjecture

$$S_n = \frac{n^3 + 3n^2 + 2n}{3}.$$

So far all we know is that the conjecture is true for $n = 0$, 1, 2, and 3. But now that we have a conjectured formula for S_n, we can verify that it is correct for all n in the same way as before.

Another Way. Here is a nice combinatorial argument. The number of ways of choosing 3 numbers from the collection $\{1, 2, 3, \ldots, n+2\}$ is

$$\binom{n+2}{3}, \qquad \text{that is,} \qquad \frac{n(n+1)(n+2)}{6}.$$

We now calculate the number of choices another way.

Imagine having chosen the 3 numbers, and let x be the largest of the 3 chosen numbers. Maybe $x = 3$. Then there are $\binom{2}{2}$ possibilities for the other two. Maybe $x = 4$. Then there are $\binom{3}{2}$ possibilities for the other two. If $x = 5$, there are $\binom{4}{2}$ possibilities for the other two. Go on in this way. Finally, if $x = n+2$ there are $\binom{n+1}{2}$ possibilities for the other two. We conclude that

$$\frac{n(n+1)(n+2)}{6} = \binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \cdots + \binom{n+1}{2}.$$

Finally, note that $\binom{k}{2} = k(k-1)/2$, and multiply by 2 to get the desired result.

Comments.

1. This result is mentioned by Āryabhata (sixth century).

2. The sum of the problem can be rewritten as

   $$(1^2 + 1) + (2^2 + 2) + \cdots + (n^2 + n).$$

   We conclude that

   $$(1^2 + 2^2 + \cdots + n^2) + \frac{n(n+1)}{2} = \frac{n(n+1)(n+2)}{3}.$$

   Now a little manipulation gives a formula for $1^2 + 2^2 + \cdots + n^2$.

3. We could turn the preceding argument on its head as follows. Look up or in some other way find the sums $1^2 + 2^2 + \cdots + n^2$ and $1 + 2 + \cdots + n$.

   The first sum turns out to be $n(n+1)(2n+1)/6$. The second sum is much easier, it is $n(n+1)/2$. This result is already seen in elementary school if the students are lucky enough to have a teacher who likes mathematics. You may know the (probably invented) story about Gauss and $1 + 2 + \cdots + 100$.

   Thus $S_n = n(n+1)(2n+1)/6 + (n)(n+1)/2$, which easily simplifies to $(n)(n+1)(n+2)/3$. This is not really a good way of finding S_n. To use this method, we need to know $1^2 + 2^2 + \cdots + n^2$, which is a less natural object than S_n.

4. The ideas used in the various arguments generalize. For example, we can show in the same way that

   $$1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \cdots + n(n+1)(n+2) = \frac{n(n+1)(n+2)(n+3)}{4}.$$

Problem 3. (a) Show how to divide the set $\{1, 2, 3, 4, \ldots, 98, 99\}$ into three subsets (not necessarily of the same size) so that the sum of the numbers in all three sets will be the same. (b) For what positive integers n can the same task be accomplished with the set $\{1, 2, 3, 4, \ldots, n-1, n\}$?

Solution. (a) We oonly describe one of the many ways to do the splitting. Let $\{a, a+1, a+2, a+3, a+4, a+5\}$ be a set of 6 consecutive integers. We can split the set into three sets with equal sums as follows: $\{a, a+5\}$, $\{a+1, a+4\}$, and $\{a+2, a+3\}$.

Our original set $\{1, 2, 3, \ldots, 99\}$ can be thought of as $\{1, 2, 3, \ldots, 9\}$ together with 15 sets of 6 consecutive integers. These 15 sets can each be split into three sets with equal sums. So we will be finished if we can show how to split $\{1, 2, 3, \ldots, 9\}$. This is easy. We can use $\{1, 2, 3, 4, 5\}$, $\{6, 9\}$, and $\{7, 8\}$.

(b) It is a standard fact that

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.$$

Thus if our set can be divided into three subsets with equal sums, then $n(n+1)/2$ must be a multiple of 3. This is the case if (i) n is a multiple of 3 or (ii) $n+1$ is a multiple of 3. We will show that, in each case, the splitting can be done unless n is "too small."

We first examamine case (i). There are two possibilities. Maybe n is even, and therefore a multiple of 6. As we observed in part (a), every set of 6 consecutive integers can be split into three parts with equal sums. Thus for any positive integer k, any set of $6k$ consecutive positive integers can be split into three parts with equal sums.

Or maybe n is odd. It is obvious that the set $\{1, 2, 3\}$ cannot be split in the desired way. We will show that if n is odd and greater than or equal to 9, our set can be split. For then $n = 9 + 6k$ for some non-negative integer k, so just as in part (a), the original set can be considered as $\{1, 2, 3, \ldots, 9\}$ together with k sets of 6 consecutive integers. But $\{1, 2, 3, \ldots, 9\}$ can be split, as can any set of 6 consecutive integers, and we are done.

Now we examine case (ii). It is obvious that $\{1, 2\}$ cannot be split. But $\{1, 2, 3, 4, 5\}$ can be, as $\{1, 4\}$, $\{2, 3\}$, and $\{5\}$. And $\{1, 2, 3, \ldots, 8\}$ can be, as $\{1, 2, 3, 6\}$, $\{5, 7\}$, and $\{4, 8\}$.

If $n+1$ is a multiple of 3 greater than 3, then $n = 5+6k$ or $n = 8+6k$ for some non-negative integer k, so our original set can be considered as $\{1, 2, 3, \ldots, 5\}$ or $\{1, 2, 3, \ldots, 8\}$ together with k sets of 6 consecutive integers, and we are done.

Problem 4. Find (with proof) the smallest possible value of $xy+yz+xz$, given that x, y, and z are real numbers such that $x^2 + y^2 + z^2 = 1$.

Solution. Any square is non-negative, so $(x+y+z)^2 \geq 0$. Expand the square:

$$(x + y + z)^2 = x^2 + y^2 + z^2 + 2xy + 2yz + 2zx.$$

Since $x^2 + y^2 + z^2 = 1$, we conclude that $1 + 2xy + 2yz + 2zx \geq 0$, and therefore $xy + yz + zx$ can never be less than $-1/2$.

To show that $xy+yz+zx$ can actually be $-1/2$, look back on the calculation and note that there is equality when $x+y+z=0$. So we need to show that there are numbers x, y, and z such that

$$x^2+y^2+z^2=1 \qquad \text{and} \qquad x+y+z=0.$$

This is evident from the geometry. The first equation is the equation of a sphere with center the origin and radius 1, while the second is the equation of a plane through the origin. The sphere and the plane meet at many places.

Alternately, we can find explicit x, y, and z that satisfy both equations. Let a, b, and c be three numbers, not all 0 but with sum 0, like 1, -1, and 0. Divide each by $\sqrt{a^2+b^2+c^2}$ and call the results x, y, and z.

Problem 5. Let $N=(3+2\sqrt{2})^{512}+(3-2\sqrt{2})^{512}$. What is the rightmost decimal digit of N?

Solution. It is maybe not even obvious that $(3+2\sqrt{2})^{512}+(3-2\sqrt{2})^{512}$ is an integer. But that will come out as a consequence of our calculations.

It is very much worthwhile to do some calculator exploration. In general let

$$C_n=(3+2\sqrt{2})^n+(3-2\sqrt{2})^n.$$

We may want to calculate C_n for a few (small) integer values of n. Of course C_{512} is very much beyond the range of an ordinary calculator.

A mathematician would probably start with $n=0$ rather than $n=1$. For one thing, C_0 is easy to find. The calculator says that $C_0=2$, $C_1=6$, $C_2=34$, $C_3=198$, $C_4=1154$, $C_5=6726$, $C_6=39202$, $C_7=228486$, $C_8=1331714$, $C_9=7761798$, and $C_{10}=45239074$.

So the calculator thinks the first few C_n are integers. Of course that does not mean that *all* the C_n are integers. It does not even mean that (for example) C_8 is an integer: the calculator only computes to a limited accuracy. So if C_8 happened to be equal to $1331714.00000000062\ldots$, the calculator would report that the answer is 1331714.

But we will not worry about this for a while. We could continue, and perhaps spot a pattern in the final digits. So far we have 2, 6, 4, 8, 4, 6, 2, 6, 4, 8, 4. It looks as if there is cycling, the pattern 2, 6, 4, 8, 4, 6 seems to repeat.

Maybe since we are asked about C_{512}, and 512 is a power of 2, we should look only at C_n where n is a power of 2. We calculate the last digits of C_1, C_2, C_4, C_8, and so on. We get digits 6, 4, 4, 4. Unfortunately my calculator now goes into scientific notation, since C_{16} is greater than 10^{12}, and gives no further information about last digits. The calculator bundled with Microsoft Windows goes a bit further: it thinks that C_{16} and C_{32} end in a 4. So there is some reason to conjecture that the final digit of C_{512} is 4, though the 6 at $n=1$ is a little unsettling.

Since we were only asked about C_{512}, and experimentation suggests that the situation when n is a power of 2 may be particularly simple, we confine attention for now to $n=2$, 4, 8, 16, and so on.

For brevity, let $u = 3 + 2\sqrt{2}$ and $v = 3 - 2\sqrt{2}$. We calculate $u^2 + v^2$ "by hand," using an algebraic trick that preserves the symmetry between u and v. Recall that for any x and y,

$$x^2 + y^2 = (x + y)^2 - 2xy. \tag{1}$$

Put $x = u$ and $y = v$. An easy calculation shows that $x + y = 6$ and $xy = 1$ (nice!). We conclude that

$$C_2 = u^2 + v^2 - 2uv = 6^2 - 2 = 34.$$

Now we calculate $u^4 + v^4$. Put $x = u^2$ and $y = v^2$. Then from Identity 1, we get

$$C_4 = u^4 + v^4 = (u^2 + v^2)^2 - 2u^2v^2 = 34^2 - 2 = 1154.$$

Note that our calculations are more reliable than the earlier ones. For they are now purely *integer* calculations, and not subject to the roundoff errors that might arise when the calculator tries to handle powers of $3 + 2\sqrt{2}$.

Similarly, we have

$$C_8 = u^8 + v^8 = (u^4 + v^4)^2 - 2u^4v^4 = (1154)^2 - 2 = 1331714.$$

In the same way, we find that

$$C_{16} = u^{16} + v^{16} = (u^8 + v^8)^2 - 2u^8v^8 = (1331714)^2 - 2.$$

A simple calculator cannot handle the square of 1331714. But it is easy to see that since 1331714 ends in a 4, its square ends in a 6, so when we subtract 2, the result ends in a 4.

Continue. In general,

$$C_{2k} = u^{2k} + v^{2k} = (u^k + v^k)^2 - 2u^kv^k = C_k^2 - 2. \tag{2}$$

Thus if C_k is an integer, then so is C_{2k}. In particular, we can conclude that C_n is an integer whenever n is a power of 2 (actually, it is an integer for all n.)

From Equation 2, we can also deal with the last digit question. For if the decimal expansion of C_k ends in a 4, the decimal expansion of C_k^2 ends in a 6, and therefore by Equation 2 the decimal expansion of C_{2k} ends in a 4.

We saw that the decimal expansion of C_2 ends in a 4. Therefore so does the decimal expansion of C_4. It follows that the decimal expansion of C_8 ends in a 4. This implies that the decimal expansion of C_{16} ends in a 4, but then so does the decimal expansion of C_{64}, of C_{128}, of C_{256}, of C_{512} (our problem), and so on forever.

Another Way. If n is not a power of 2, things get more complicated. We got to 512 very quickly, by repeated doubling. We will explore a slower and more painful path, because it will have some interesting surprises.

Instead of going C_2, C_4, C_8, C_{16}, $\ldots$, we look at C_2, C_3, C_4, C_5, $\ldots$. It would be useful to express C_{n+1} in terms of C_n. We will not quite succeed, but still will come up with a useful result.

By multiplying out, we can see that in general

$$(x^n + y^n)(x + y) = x^{n+1} + y^{n+1} + xy(x^{n-1} + y^{n-1})$$

or equivalently

$$x^{n+1} + y^{n+1} = (x^n + y^n)(x + y) - xy(x^{n-1} + y^{n-1}). \tag{3}$$

Let $x = u$ and $y = v$. Then $x+y = 6$, $xy = 1$, $x^{n+1}+y^{n+1} = C_{n+1}$, $x^n+y^n = C_n$, and $x^{n-1} + y^{n-1} = C_{n-1}$. Then Identity 3 yields the following recurrence:

$$C_{n+1} = 6C_n - C_{n-1}. \tag{4}$$

The above recurrence is structurally similar to the recurrence $F_{n+1} = F_n + F_{n-1}$ that we saw in connection with the *Fibonacci Sequence* (see December 2005, January 2006).

An easy calculation shows that $C_0 = 2$ and $C_1 = 6$. Now use Recurrence 4 to calculate. We get $C_2 = 34$, $C_3 = 198$, $C_4 = 1154$, $C_5 = 6726$, $C_6 = 39202$. The calculations are easy, and could be continued for a while. But fairly quickly we bump into the limitations of the calculator.

However, Recurrence 4 does settle the question of whether C_n is an integer. Certainly C_0 and C_1 are. But $C_2 = 6C_1 - C_0$, so C_2 is an integer. But $C_3 = 6C_2 - C_1$, so C_3 is an integer. But $C_4 = 6C_3 - C_2$, so C_4 is an integer. It is clear that this chain of reasoning can be carried arbitrarily far: we conclude that C_n is an integer for every non-negative integer n.

Recurrence 4 also lets us deal with "last digit" questions. In general, let D_n be the last digit of C_n. Then $D_0 = 2$, $D_1 = 6$, $D_2 = 4$, and $D_3 = 8$. (This one is a little tricky. When we subtract C_1, which has last digit 6, from $6C_2$, which has last digit 4, the last digit in the result is 8: just think about the ordinary subtraction process.)

Continue. We get $D_4 = 4$, $D_5 = 6$, $D_6 = 2$, $D_7 = 6$, $D_8 = 4$, $D_9 = 8$, $D_{10} = 4$, $D_{11} = 6$, $D_{12} = 2$, $D_{13} = 6$. We sum up the first few results:

$$2, \quad 6, \quad 4, \quad 8, \quad 4, \quad 6, \quad 2, \quad 6, \quad 4, \quad 8.$$

Note that $D_0 = 2$, $D_1 = 6$, and $D_6 = 2$, $D_7 = 6$. But in general the value of C_{n+1} is completely determined by the values of C_n and C_{n-1}, and therefore D_{n+1} is determined by D_n and D_{n-1}. Thus the sequence D_0, D_1, D_2, and so on is periodic with period 6. (We noted the apparent pattern 2, 6, 4, 8, 4, 6, 2, 6, 4, ... when we used the calculator to find the first few C_n. But now we have a proof.)

It is now easy to find D_{512}. Since 6 divides 510, another period starts at $n = 510$. Thus $D_{510} = 2$, $D_{511} = 6$, and $D_{512} = 4$.

Comment. Note that in general

$$(a + b\sqrt{k})(c + d\sqrt{k}) = ab + kbd + (ac + bd)\sqrt{k} \qquad \text{and}$$
$$(a - b\sqrt{k})(c - d\sqrt{k}) = ab + kbd - (ac + bd)\sqrt{k}.$$

By applying the above equations repeatedly, we can see that if a and b are integers, and k and n are positive integers, then

$$(a+b\sqrt{k})^n = A_n + B_n\sqrt{k} \qquad \text{and}$$
$$(a-b\sqrt{k})^n = A_n - B_n\sqrt{k}$$

where A_n and B_n are integers. In particular, $(a+b\sqrt{k})^n + (a-b\sqrt{k})^n = 2A_n$, an integer, indeed an even integer.

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