Solutions to December 2009 Problems

Problem 1. Alphonse and Beth play the following game. A neutral third party alternately tosses a biased penny, which has probability 2/3 of landing heads, and a biased dime, which has probability 1/3 of landing heads. The penny is tossed first. The game ends when two consecutive heads first occur, in which case Alphonse wins, or when two consecutive tails first occur, in which case Beth wins. What is the probability that Alphonse wins?

Solution. It is convenient to write H for head on the penny, T for tail on the penny, h for head on the dime, and t for tail on the dime.

We look first at patterns that start with H (a head on the penny), and in which Alphonse wins. They are Hh (head on the penny followed by a head on the dime), HtHh, HtHtHh, and so on. The probability that the first toss is a head is 2/3. Given this, the probability that the second toss is a head is 1/3. So the probability of Hh is (2/3)(1/3). A similar calculation shows that the probability of HtHh is (2/3)(2/3)(2/3)(1/3), and that the probability of HtHtHh is (2/3)(2/3)(2/3)(2/3)(2/3)(1/3). Continue. We find that the probability that the first toss is H *and* Alphonse wins is

$$\frac{2}{9}\left[1+\frac{4}{9}+\left(\frac{4}{9}\right)^2+\left(\frac{4}{9}\right)^3+\cdots\right].$$

The standard formula for the sum of a geometric progression gives that the above expression is equal to 2/5.

Now we look at patterns that start with T, and in which Alphonse wins. They are ThH, ThThH, ThThThH, and so on. The associated probabilities are (1/3)(1/3)(2/3), (1/3)(1/3)(1/3)(1/3)(2/3), and so on. So the probability that the first toss is T *and* Alphonse wins is

$$\frac{2}{27}\left[1+\frac{1}{9}+\left(\frac{1}{9}\right)^2+\left(\frac{1}{9}\right)^3+\cdots\right].$$

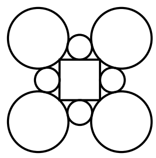
The above sum is equal to 1/12. So the probability that Alphonse wins is 2/5 + 1/12, that is, 29/60. Maybe this is a bit of a surprise. It turns out that Beth has a somewhat higher probability of winning than does Alphonse. One might have guessed that since the penny favours Alphonse, and the penny is tossed first, Alphonse would have an advantage. Intuition in probability questions can all too often be wrong.

Comment 1. We can generalize the problem a bit. Suppose that the probability of a head on the penny is P, and that the probability of a head on the dime is p. Then the analogue of our first infinite sum is

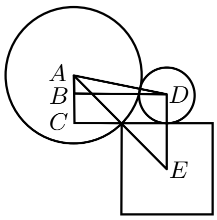
$$Pp + P(1-p)Pp + (P(1-p))^2 + Pp(P(1-p))^3 + \cdots ,$$

an easily summed infinite geometric series. It is just as easy to write down and evaluate an analogue of our second infinite sum.

Problem 2. In the figure below, we have a square, and four congruent circles (large in the picture) that pass through the vertices of the square, and are symmetric with respect to the axes of symmetry of the square. We also have four congruent circles (small in the picture), which are tangent to various lines and "large" circles as shown. Given that "large" circles have radius a, and "small" circles have radius b, what is the side of the square?



Solution. Let A, B, C, D, and E as in the picture below. We eliminated some of the circles of the original problem to simplify things. In the picture, A and D are the centers of two adjacent circles, and E is the center of the square. (The centers are almost always useful in problems about circles.) Note that in the picture, lines that look parallel *are* parallel, and lines that look perpendicular *are* perpendicular.



The line AE makes $45°$ angles with the sides of the square. It follows that $AC = a/\sqrt{2}$. But $BC = b$, and therefore $AB = a/\sqrt{2} - b$.

Note that $BD = a/\sqrt{2} + s/2$ and $AD = a + b$. Thus, using the Pythagorean Theorem on $\triangle ABD$, we obtain

$$\left(\frac{a}{\sqrt{2}} - b\right)^2 + \left(\frac{a}{\sqrt{2}} + \frac{s}{2}\right)^2 = (a+b)^2.$$

Expand and simplify. After a while we reach the equation

$$s^2 + 2\sqrt{2}as - 4(2+\sqrt{2})ab = 0.$$

Solve for s, discarding the negative solution. We find that

$$s = \sqrt{2a^2 + 4(2+\sqrt{2})ab} - \sqrt{2}a.$$

This is the end of the story, at least if the "large" circles are indeed substantially larger than the "small" circles. Recall that we wrote $AB = a/\sqrt{2} - b$. This does not make sense if $a/\sqrt{2} < b$, and indeed the picture above is obviously wrong in that case. But for $b > a/\sqrt{2}$, one can draw a suitable analogous picture, and use the Pythagorean Theorem in a very similar way. After a while, we obtain the same quadratic equation for s as the one obtained above, with of course the same solution.

Another Way. We can use a much less elaborate diagram. Put in the three centers A. D, and E, and form $\triangle AED$. Note that $\angle AED = 45°$. The sides of $\triangle AED$ are easy to compute: $AD = a + b$, $EA = s/\sqrt{2} + a$, and $ED = s/2 + b$. Thus by the Cosine Law,

$$(a+b)^2 = \left(\frac{s}{\sqrt{2}} + a\right)^2 + \left(\frac{s}{2} + b\right)^2 - \frac{1}{\sqrt{2}}\left(\frac{s}{\sqrt{2}} + a\right)\left(\frac{s}{2} + b\right).$$

Expand and simplify. After a while, we reach the quadratic equation that was reached in the first solution. A pleasant thing about this approach is that there is no need for two separate diagrams, no need to worry about how large is "large."

Problem 3. Find all solutions in real numbers of the following system of equations:

$$\begin{aligned} x(y^2 + z^2) &= 11yz, \\ y(z^2 + x^2) &= 12xz, \\ z(x^2 + y^2) &= 13xy. \end{aligned}$$

Solution. First we deal with trivial cases. If, for example, $x = 0$, then the second and third equations become $yz^2 = 0$ and $zy^2 = 0$, so at least one of y and z is 0. Conversely, it is easy to see that putting two of the variables equal to 0, and letting the other roam freely, gives a solution. Thus we have the trivial solutions $(0, 0, t)$, $(0, t, 0)$, and $(t, 0, 0)$, where t is arbitrary.

Now look for solutions with none of x, y, or z equal to 0. Then by dividing both sides of the first equation by yz, both sides of the second by xz, and both sides of the third by xy, we obtain the system

$$\begin{aligned} \frac{xy}{z} + \frac{zx}{y} &= 11, \\ \frac{yz}{x} + \frac{xy}{z} &= 12, \\ \frac{zx}{y} + \frac{yz}{x} &= 13. \end{aligned}$$

Let $w = xy/z$, $v = zx/y$, and $u = yz/x$. Our system becomes $w + v = 11$, $u + w = 12$, $v + u = 13$. Add. We get $2(u + v + w) = 36$, so $u + v + w = 18$. It follows that $u = 7$, $v = 6$, and $w = 5$.

So $xy/z = 5$, $zx/y = 6$, and $yz/x = 7$. Multiply. We get $xyz = 210$. From this and $xy/z = 5$, we get, by dividing, that $z^2 = 42$. Similarly, $y^2 = 35$ and $x^2 = 30$.

Thus the only solution with the variables positive is $(\sqrt{30}, \sqrt{35}, \sqrt{42})$. From this solution we can get three other solutions by putting a minus sign in front of *exactly two* of our entries.

Problem 4. Find all positive integer solutions of $x^2 - xy - y^2 = \pm 1$. (A proof that you have found them all is needed.)

Solution. We look for positive integers x, y such that (x, y) is a solution of the equation $|x^2 - xy - y^2| = 1$. It is easy to see that $(x, y) = (1, 1)$ is a solution (here $x^2 - xy - y^2 = -1$), and that $(x, y) = (2, 1)$ is also a solution (here $x^2 - xy - y^2 = 1$). Note also that $(x, y) = (3, 2)$ and $(x, y) = (5, 3)$ are also solutions. It is a bit of a leap, but we may guess that the consecutive Fibonacci number pairs $(8, 5)$ and $(13, 8)$ are also solutions. Indeed they are.

Recall that the Fibonacci numbers F_k are defined by $F_0 = 0$, $F_1 = 1$, and $F_{k+2} = F_{k+1} + F_k$ for all $k \geq 0$. (There are other not quite equivalent definitions for the Fibonacci sequence.) The numerical evidence obtained so far may lead us to conjecture that the ordered pair (F_{k+1}, F_k) always is a solution. We need to do two things: (i) show that this is indeed true, and (ii) show that there are no other solutions. The first task is relatively straightforward; the second is more delicate.

(i) Let the ordered pair $(x, y) = (u, v)$ be a solution of the equation $|x^2 - xy - y^2| = 1$. We show that the ordered pair $(u + v, u)$ is also a solution. Simple substitution shows that

$$(u + v)^2 - (u + v)u - u^2 = -u^2 + uv + v^2 = -(u^2 - uv - v^2).$$

Since (u, v) is a solution of $|x^2 - xy - y^2| = 1$, it follows that

$$|(u + v)^2 - (u + v)u - u^2| = 1,$$

and therefore $(u + v, u)$ also is a solution.

Suppose that we know that (F_{k+1}, F_k) is a solution. Then by the result above, $(F_{k+1} + F_k, F_{k+1})$ is a solution, that is, (F_{k+2}, F_{k+1}) is a solution. So *because* $(1, 1)$ is a solution, $(2, 1)$ is a solution. But *because* $(2, 1)$ is a solution, so is $(3, 2)$. But because $(3, 2)$ is a solution, so is $(5, 3)$, and because of that, so is $(8, 5)$, and so on. We conclude that for every positive integer k, (F_{k+1}, F_k) is a solution.

(ii) We now show that there are no other positive solutions. The main idea is not very different from that of (i), except that we go backwards. Suppose that u and v are (real) numbers such that the ordered pair $(x, y) = (u, v)$ is a solution of $|x^2 - xy - y^2| = 1$. We show that the ordered pair $(v, u - v)$ is also a solution. This is a straightforward calculation, for it is easy to verify that

$$v^2 - v(u - v) - (u - v)^2 = -(u^2 - uv - v^2).$$

Now start with a solution (x, y) in positive integers. It is easy to verify that $x \geq y$, with equality only if $x = 1$. So if $x \geq 2$, then $(y, x - y)$ is also a solution in positive integers, and the first component y of this solution is less than the first component x of (x, y).

Continue, doing with $(y, x - y)$ what we did with (x, y): if $y \geq 2$, we get a new positive solution, namely $(x - y, 2y - x)$. Continue, as long as the first component is ≥ 2. We get a sequence of solutions, with first component steadily decreasing. This cannot go on forever. So after a while, we reach an ordered pair (u, v) of positive integers which is a solution, with $u \geq 2$, such that $(v, u - v)$ is a positive solution but $v < 2$. That means that $v = 1$, and therefore $u - v = 1$.

So by descending from a positive solution (x, y) with $x \geq 2$, after a while we reach the solution $(1, 1)$. The solution "just before" $(1, 1)$ in the descent process must have been $(2, 1)$, since the descent process always replaces (s, t) by $(t, s - t)$, and if $t = 1$ and $s - t = 1$ then $s = 2$ and $t = 1$. Similarly, the solution "just before"' $(2, 1)$ in the descent process must have been $(3, 2)$, and so on. Climbing in this way, we find that the original (x, y) must have been of the shape (F_{k+1}, F_k) for some k. This proves that no ordered pair (x, y) of positive integers satisfies our equation except for the ordered pairs (F_{k+1}, F_k), where k is a positive integer.

http://www.pims.math.ca/education/math_problems/
http://www.math.ubc.ca/~adler/problems/