Solutions to October 2008 Problems

Problem 1. Solve for x: $\dfrac{x-a}{b}+\dfrac{x-b}{a}=\dfrac{b}{x-a}+\dfrac{a}{x-b}$.

Solution. Here we take a and b to be unspecified constants. Note that neither a nor b can be 0, and x cannot be equal to a or to b. With these provisos, our equation can be rewritten as

$$\frac{(a+b)x-(a^2+b^2)}{ab}=\frac{(a+b)x-(a^2+b^2)}{(x-a)(x-b)}. \tag{1}$$

We have equality if $(a+b)x-(a^2+b^2)=0$. This gives the "solution" $x=(a^2+b^2)/(a+b)$. Note for the future that this is a solution only if $a+b\neq 0$.

If $(a+b)x-(a^2+b^2)$ is not equal to 0, then Equation 1 is equivalent to

$$\frac{1}{ab}=\frac{1}{(x-a)(x-b)}, \tag{2}$$

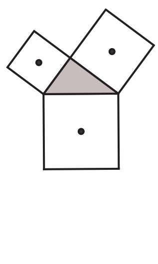
which after some simplification is equivalent to $x^2-(a+b)x=0$. This has the solutions $x=0$ and $x=a+b$.

Thus if neither a nor b is 0, and $b\neq -a$, the solutions are given by $x=(a^2+b^2)/(a+b)$, $x=0$, and $x=a+b$. If $b=-a$, the only solution is $x=0$.

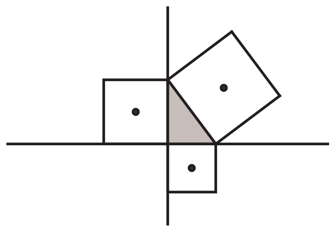
Problem 2. Every point on the x-axis is coloured black or white. Show that there are two distinct points X, Y such that X, Y, and the point halfway between X and Y all have the same colour.

Solution. We won't bother to draw a picture, but we should draw one while reading through the argument. If all points, or all but one point, are white, the result is obvious. Now take two points that are coloured black, say A and B. If the point C halfway in between is black, we are finished. So assume C is white. Take the point D to the left of A, with $DA=AB$. Clearly D must be white, or we are finished. The same is true for E, to the right of B, with $EA=AB$. But then C is white and is halfway between the white points D and E.

Problem 3. Outward facing squares are erected on the sides of a right-angled triangle whose legs have length a and b. What is the area of the triangle whose vertices are the midpoints of these squares?



Solution. The problem may feel easier if we draw the picture a little differently. Bigger would be nice. But it is also good to rotate the diagram as in the picture below. Now draw in the usual x and y axes. So the vertex with the right angle is at the origin, and the other two vertices are at $(a, 0)$ and $(0, b)$.



We compute the coordinates of the centres of the various squares. Two of these are very easy. The square on the side that joins $(0, 0)$ to $(a, 0)$ has centre with coordinates $(a/2, -a/2)$, while the square on the side that joins $(0, 0)$ to $(0, b)$ has centre with coordinates $(-b/2, b/2)$.

Now look at the square on the hypotenuse. The vertex of this square that is nearest $(a, 0)$ is obtained from $(a, 0)$ by going to the right by an amount b, and up by an amount a. So its coordinates are $(a+b, a)$. The centre of the square is the midpoint of the diagonal that joins $(a + b, a)$ to $(0, b)$. Thus its coordinates are $((a + b)/2, (a + b)/2)$.

Let ℓ be the line that joins the origin to the centre of the square on the hypotenuse. By the calculation of the preceding paragraph, line ℓ has slope 1, so it bisects the right angle. This solves a rather commonly asked question. And as a bonus we find that the line that joins the centres of the other two squares goes through the origin, and is perpendicular to ℓ.

We are almost finished. Our rectangle can be viewed as having as "'base" the line segment that joins $(-b/2, b/2)$ to $(a/2, -a/2)$ and as height the line segment that joins the point $((a + b)/2, (a + b)/2)$ to the origin. Both base and height have length $(a + b)/\sqrt{2}$, so the required area is $(a + b)^2/4$.

Comment. We give a related result which is maybe a little surprising. Let $WXYZ$ be a quadrilateral, where the vertices have been named in counter-clockwise order. Erect squares on sides WX, XY, YZ, and ZW such that the square on AB is on the right as you travel from W towards X, with similar conditions on the squares on the other sides (if the polygon were convex, one could call them outward facing squares). Let K, L, M, and N be, in order, the centre of these squares. Then KM and LN are perpendicular, and of the same length. This result applies to our triangle also, for we can view it as a degenerate quadrilateral with two vertices at the right angle, and a square of side 0 erected there.

We give a crude proof of the above result. A nicer version would use vector notation, or even better identify the vertices with complex numbers. And (maybe) a nicer proof still would not use coordinates at all. Let the coordinates of W, X, Y, Z be (a, b), (c, d), (e, f), and (g, h). We will find the coordinates of K, L, M, and N.

First let O be the origin, and let P have coordinates $(q, r))$. We find the coordinates of the centre of the square on OP which is on the right as we travel from O to P. First we find the coordinates of the vertex T of the square (other than P) which is nearest to O. We want OT to be perpendicular to OP and of the same length. There are two candidates for the coordinates of T, namely $(-q, p)$ and $(p, -q)$. A little thinking shows that $(-q, p)$ is the right one. Now the centre of the square is easily seen to be the midpoint of TP, which is $((-q+p)/2, (p+q)/2$.

Next we find the coordinates of K. First we pull back (a, b) to the origin (subtract a from first coordinate, b from the second. Do the same thing to (c, d)). Now we have a "origin and other point" situation. Compute the appropriate corner of the square, then find the centre. Then add a to the x-coordinate, b to the y coordinate to undo the shift we did earlier. After some work, we find that the coordinates of K, L, M, and N are:

$$\left(\frac{c-d+a+b}{2}, \frac{c+d-a+b}{2}\right), \qquad \left(\frac{e-f+c+d}{2}, \frac{e+f-c+d}{2}\right),$$
$$\left(\frac{g-h+e+f}{2}, \frac{g+h-e+f}{2}\right), \qquad \left(\frac{a-b+g+h}{2}, \frac{a+b-g+h}{2}\right).$$

Now calculate the difference between the coordinates of K and M, and of L and N. The results are of the form (x, y) and $(y, -x)$, which completes the proof.

Problem 4. Find all integers n such that $n + 16$ is a perfect square and no prime greater than 3 divides n. (A correct list is not sufficient: one needs to show that there are no others.)

Solution. We want to solve the equation $n + 16 = x^2$, with the proviso that x and n are integers, and no prime greater than 3 divides n. Without loss of generality we may assume that $x \geq 0$.

Rewrite out equation as $n = (x - 4)(x + 4)$. It is convenient to examine first the cases where $0 \leq x \leq 4$, that is, cases where n is non-positive. It is easy to check that the only situations where no prime greater than 3 divides n correspond to $x = 0$ and $x = 2$, giving the answers $n = -16$ and $n = -12$.

So from now on we can assume that $x > 4$. It is convenient to let $y = x - 4$. So we are looking at the equation $n = y(y + 8)$, where y is positive.

We deal first with the case where y is odd. Then y and $y + 8$ must each be powers of 3. That forces $y = 1$, giving the answer $n = 9$.

Next we deal with the possibility that y is divisible by 2, but by no higher power of 2. So $y = 2s$ where s is odd. We now want $4s(s + 4)$ to have no prime divisor greater than 3. This forces s and $s + 4$ each to be a power of 3, which is impossible.

Next we deal with the possibility that y is divisible by 4, but by no higher power of 2. So $y = 4t$ where t is odd. So we now want $16t(t + 2)$ to have no prime divisor greater than 3. This forces $t = 1$, giving the answer $n = 48$.

Next we deal with the possibility that y is divisible by 8 but by no higher power of 2. So let $y = 4u$ where u is odd. So we want $64u(u + 1)$ to have no prime divisor greater than 3. Since u is odd, we must have $u = 3^a$ for some non-negative integer a. It is clear that $u + 1$ cannot be divisible by 3, so $u + 1 = 2^b$ for some positive integer b. We therefore want to have $3^a + 1 = 2^b$. This has two obvious solutions: $a = 0$, $b = 1$, and $a = 1$, $b = 2$. That gives the answers $n = 128$ and $n = 768$. We will show that $3^a + 1 = 2^b$ has no other solutions.

Let $b > 2$. Then 2^b is a multiple of 8. But it is easy to see that $3^a + 1$ can never be a multiple of 8. For if a is even, then 3^a is a power of 9, and therefore leaves a remainder of 1 on division by 8, so $3^a + 1$ leaves a remainder of 2 on division by 8. If a is oddd, then 3^a leaves a remainder of 3 on division by 8, so $3^a + 1$ leaves a remainder of 4 on division by 8. Thus if $b > 2$ then we cannot have $3^a + 1 = 2^b$.

Finally, we deal with the possibility that y is divisible by a power of 2 greater than 8. Let $y = 2^{c+3}v$ where v is odd. We want $(2^{c+3}v)(8)(2^c v + 1)$ to have no prime divisor greater than 3. Since v is odd, it must be a power of 3. But $2^c v + 1$ is also odd, so it also must be a power of 3. It is clear that if 3 divides v, then 3 cannot divide $2^c v + 1$. It follows that $v = 1$. Thus we want $2^c + 1$ to be a power of 3.

This happens if $c = 1$ and also if $c = 3$. The case $c = 1$ gives the answer $n = 384$, while $c = 3$ gives the answer $n = 4608$.

To complete our argument, we need to show that $2^c + 1 = 3^d$ has no solutions in integers apart from $c = 1$ and $c = 3$. Let $c \geq 2$. Then $2^c + 1$ leaves a remainder of 1 on division by 4. But 3^d leaves a remainder of 1 on division by 4 only if d is even. It follows that if $2^c + 1 = 3^d$, then d must be even, say $d = 2e$. Thus $3^d - 1 == 3^{2e} - 1 = (3^e - 1)(3^e + 1) = 2^d$. It follows that $3^e - 1$ and $3^e + 1$ are each powers of 2. But these two numbers differ by 2. The only way that this can happen is if $3^e - 1 = 2$, that is, if $d = 2$ (and therefore $c = 3$).

Comment. To deal with the last two cases, we examined the solutions in (non-negative) integers of the equation $3^x - 2^y = \pm 1$. There is a famous more general old problem. In 1844, Catalan conjectured that if a, b, x, and y are integers all greater than 1, then the only solution to the equation $x^a - y^b = 1$ is given by $x = 3$, $a = 2$, $y = 2$, $b = 3$. Catalan's conjecture remained unsolved for more than a century and a half, though over the years a good deal of information was amassed. In particular, Tijdeman (1974) made very considerable progress. Finally, in 2002, Mihăilescu proved the full Catalan conjecture. This is one more example of an old conjecture that has only been settled quite recently. It thus joins the substantially more celebrated Fermat's Last Theorem.