Solutions to October 2006 Problems

Problem 1. Find exact expressions for the real solutions of the equation

$$(x+1)(x+2)(x+3)(x+4) = 99.$$

Solution. A start along the following lines seems to be the most popular. The equation holds if and only if

$$(x+1)(x+4)(x+2)(x+3) = 99,$$

or equivalently if and only if

$$(x^2+5x+4)(x^2+5x+6) = 99. \tag{1}$$

Let $y = x^2+5x+5$. Then Equation (1) holds if and only if

$$(y-1)(y+1) = 99,$$

or equivalently if and only if $y^2 = 100$, that is, if and only if $y = 10$ or $y = -10$.

Now $y = 10$ if and only if $x^2+5x+5 = 10$, or, equivalently, $x^2+5x-5 = 0$. By the Quadratic Formula, this is the case if and only if $x = (-5 \pm \sqrt{45})/2$.

Finally, $y = -10$ if and only if $x^2+5x+15 = 0$. Note that

$$x^2+5x+15 = (x+5/2)^2 + (15-25/4)$$

so the right-hand side is greater than or equal to $15-25/4$ for any real x, and in particular cannot be 0 for any real x. More informally, the equation has only the (non-real) solutions $x = (-5 \pm \sqrt{-35})/2$.

It follows that the real solutions of the original equation are given by

$$x = \frac{-5 \pm \sqrt{45}}{2}.$$

We could write $3\sqrt{5}$ in place of $\sqrt{45}$. Or not.

Comment. It is tempting to note that x^2+5x+6 is 2 more than x^2+5x+4, and "because" $99 = 9 \cdot 11$, and 11 is 2 more than 9, "it follows" that $x^2+5x+4 = 9$. This is wrong, or at least incomplete. It is certainly true that if $x^2+5x+4 = 9$, then x is a solution of the original equation, and as a matter of fact in this way we get the two real solutions of that equation. However, without further work we *cannot know* that there are no other real solutions.

Instead of letting $y = x^2+5x+5$, we could have let $y = x^2+5x+4$. That gives $y(y+2) = 99$, or equivalently $y^2+2y-99 = 0$, which easily yields $y = 9$ or $y = -11$. But it is better (more symmetrical) to let $y = x^2+5x+5$.

Another Way. The collection $\{1, 2, 3, 4\}$ is symmetrically distributed around $5/2$. Let $z = x - 5/2$. Then our equation can be rewritten as

$$(z-3/2)(z-1/2)(z+1/2)(z+3/2) = 99,$$

or equivalently as

$$(z^2-1/4)(z^2-9/4) = 99.$$

But $5/4$ is midway between $1/4$ and $9/4$, so if we let $w = z^2 - 5/4$, we can rewrite the above equation as

$$(w-1)(w+1) = 99, \qquad \text{or equivalently} \qquad w^2 = 100.$$

Thus the original equation can be rewritten as $w = \pm 10$. Since $z^2 = 5/4 + w$, we can rewrite $w = \pm 10$ as

$$z^2 = 5/4 \pm 10.$$

If x is to be real, z must also be. It follows that $z = \pm\sqrt{5/4 + 10}$. Finally, add 5/2 to these z to obtain the possible real values of x.

Comment. In both solutions, though rather more explicitly in the second one, we took advantage of the symmetries in the problem. The same approach works for any equation of the shape

$$(x + a)(x + b)(x + c)(x + d) = k,$$

where a, b, c, and d are consecutive terms of an arithmetic sequence, or more generally have a center of symmetry, as in 1, 5, 7, 11.

It would be a mistake to "simplify" the original equation, by multiplying out. After some mildly tedious (and error-prone) work, we would arrive at an equation of the shape

$$x^4 + a_1x^3 + a_2x^2 + a_3x + a_4 = 0, \tag{2}$$

where, for the record, $a_1 = 10$, $a_2 = 35$, $a_3 = 50$, and $a_4 = -75$. The solutions of this equation are not at all obvious. The reason that this approach leads to difficulties is that multiplying out destroys symmetry, the "simplified" equation has less structure than the original equation. The moral of the story is that symmetry is your friend, take advantage of it. (That didn't come out sounding quite right.)

Now in fact there *is* an algebraic way of solving general quartic (degree 4) equations like Equation (2). The procedure was first found in the middle of the sixteenth century by Cardano and Ferrari, more or less as a byproduct of the solution of the general cubic equation by del Ferro, Tartaglia, and Cardano. (It is an interesting story; Cardano's life would make a wildly implausible movie.)

The procedure for solving the general quartic was improved in various ways by a number of people, including such heavyweights as Descartes, Newton, Euler, and Lagrange. But even the nicest such procedures are quite messy. (We must admit, however, that in this particular case the "standard" Cardano-Ferrari way of solving quartics turns out pleasantly.)

Comment. The Quadratic Formula is an important tool. Here is a detailed proof, more or less the standard one, but somewhat better. We want to solve the equation

$$ax^2 + bx + c = 0, \tag{3}$$

where a, b, and c are (say) real numbers and $a \neq 0$. Equation (3) is equivalent to

$$4a^2x^2 + 4abx + 4ac = 0. \tag{4}$$

We *multiplied* each side of Equation (3) by $4a$. The usual argument *divides* each side by a. That is less attractive, division is less pleasant than multiplication. Now $4a^2x^2 + 4abx$ is "almost" the same as $(2ax + b)^2$. More precisely, Equation (4) is equivalent to

$$(2ax + b)^2 = b^2 - 4ac. \tag{5}$$

If $b^2 - 4ac < 0$, then the equation has no real solution, since the square of a real number cannot be negative. If $b^2 - 4ac \geq 0$, then Equation (5) is equivalent to

$$, 2ax + b = \pm\sqrt{b^2 - 4ac} \tag{6}$$

which in turn is equivalent to

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{7}$$

In fact, the solution given in Equation (7) is *formally* correct even when $b^2 - 4ac < 0$. But for it to make full sense, we would have to develop a theory of complex numbers. Actually, a fair bit of work needs to be done even in the case $b^2 - 4ac \geq 0$, since we need to develop the theory of real numbers in order to show, for example, that every non-negative real number has a square root. But enough for now!

Problem 2. In a mathematics competition, the top five people get medals (gold, silver, bronze, plastic, cardboard). Medals were given to the five people, but because of an unfortunate mixup they were given out at random. What is the probability that exactly two of the people got the right medal?

Solution. Imagine that the top five people are lined up on the stage to receive their medals, with the person who should receive the cardboard medal on the extreme left, the person who should receive the plastic medal immediately to her left, and so on. Imagine that the person presenting the medals goes drunkenly from left to right, handing out the medals at random.

The presenter can give the leftmost person any one of 5 medals. For *every* way that she hands a medal to the leftmost person, she can hand any one of 4 medals to the next person. So there are $5 \cdot 4$ different ways of giving medals to the two leftmost people. For every such way, there are 3 ways of handing a medal to the next person. So there is a total of $5 \cdot 4 \cdot 3$ ways of handing medals to the three leftmost people.

By a similar argument, we can see that there are $5 \cdot 4 \cdot 3 \cdot 2$ ways of handing medals to the four leftmost people. Now the job is done: once we have handed out four medals, there is only 1 way of giving a medal to the remaining person. So there are

$$5 \cdot 4 \cdot 3 \cdot 2 \cdot 1,$$

that is, 120 ways of handing the medals to the five people. The final multiplication by 1 does nothing, but (to me) it makes the expression look nicer.

Since the medals were handed out "at random," each of the 120 ways of handing out the medals is *equally likely.*

Let w be the number of ways that the medals can be handed out so that exactly two people get the right medal. Then the required probability will be equal to $w/120$. So if we can find w, the problem will be solved.

We can find w by making an explicit list of all the ways that exactly two people get the right medal. This is not completely pleasant, but it is a perfectly legitimate way of settling the problem. We can, however, use a shortcut to do the counting.

Which two people get the right medal? By definition, the number of ways of *choosing* 2 people from 5 is ${}_5\mathrm{C}_2$. There are other names for the number of choices, including $\mathrm{C}(5,2)$ and C^5_2. The standard notation among mathematicians for the number of choices is $\binom{5}{2}$.

Given that a certain two people get the right medal, we count the number of ways that none of the other three get the right medal. This is the number of ways of arranging the three numbers 1, 2, and 3 so that none of them is in the "right" position. We can make an explicit list of the arrangements that qualify: $(2,3,1)$ and $(3,1,2)$.

We conclude that

$$w = 2\binom{5}{2}.$$

A standard calculation shows that $\binom{5}{2} = 10$, and therefore $w = 20$, and the required probability is $20/120$, or, if you prefer, $1/6$.

Problem 3. Alicia started going up the Grouse Grind trail at 4:30. Fred and Janet started 30 minutes later. Janet passed Alicia halfway up the Grind, and Fred passed Alicia 16 minutes afterwards. Janet got to the top 12 minutes before Fred. Everyone climbed at unvarying speed. At what time did Alicia reach the top?

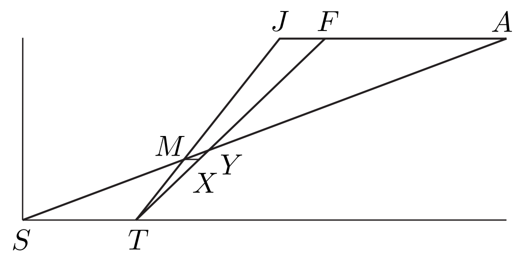
Solution. Alicia takes 30 minutes more than Janet to get to the halfway point, and therefore 60 minutes more for the full hike. But Janet takes 12 minutes less than Fred, so Alicia takes 48 minutes more than Fred.

Fred got to the halfway point 6 minutes after Janet did. When Fred caught up to Alicia, she had climbed for 16 minutes beyond the halfway point while Fred had climbed for 10. Thus Alicia's Grind time is Fred's time multiplied by $16/10$. But the difference between their times is 48 minutes, so six-tenths of Fred's time is 48 minutes. It follows that Fred's time is 80 minutes and Alicia's is 128 minutes. She arrived at 6:38.

Another Way. What variables should we introduce? Maybe the speeds of the participants. Or maybe (better) their Grind hike times. Let these be a, f, and j. We are told that $f - j = 12$. Also, Janet takes 30 minutes less than Alicia to do half the Grind, so $a - j = 60$.

Alicia's speed is $1/a$ Grinds per minute, so when Fred gets to the halfway point Alicia has a lead of $6/a$. Their relative speeds are $1/f - 1/a$, and therefore $(1/f - 1/a)(16) = 6/a$. If we simplify, we get $16a = 10f$. We now have three linear equations in three unknowns, and the rest is easy.

Another Way. The same ideas can be carried out pictorially. We compute exactly, but alternately the picture can be drawn accurately and an approximate answer obtained by measuring. The horizontal axis represents time and the vertical axis distance. The point S represents the space-time position of Alicia at the start of

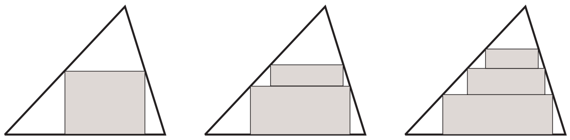


her climb, and T is the space-time position of Fred and Janet when they started. The points J, F, and A represent the space-time positions of the three climbers at the end of their climb, and M is the event "Janet passes Alicia." Because M is midway in vertical distance between S and A, the triangles MST and MAJ are congruent. In particular, since the time between S and T is 30 minutes, so is the time between J and A. We are given that $JF = 12$, so $FA = 18$.

The line segment MX is $1/2$ of JF, so $MX = 6$. Since $\triangle MXY$ is similar to $\triangle AFY$, we conclude that the vertical distance between M and Y is $1/3$ of the vertical distance between Y and A, so the vertical distance between M and Y is $1/8$ of the total Grind distance. It took Alicia 16 minutes to cover this, so the whole Grind took her 128 minutes.

Comment. The first solution shows that it is possible to argue "rhetorically," without algebra. But that may require too much concentration. A nice thing about algebra is that an equation, once written down, serves as a permanent written encapsulation of the idea that led to it, freeing up brain space for further thinking. And equations, once obtained, may have a *shape* that suggests a solution method.

Problem 4. In the figure below, each of the three large triangles has area 1, the base angles are each less than or equal to a right angle, and the shapes that look like rectangles *are* rectangles. What is the largest possible shaded area in (i) the left-hand picture; (ii) the middle picture; (iii) the right-hand picture?



Solution. (i) Look at the left-hand picture. The big triangle has been divided into four parts: (a) the top triangle; (b) the small triangle on the left; (c) the small triangle on the right; (d) the shaded rectangle.

Let the base of the top triangle be t times the base of the large triangle. The top triangle is similar to the big triangle. Indeed, it is the big triangle with all sides scaled by a scale factor of t. But scaling linear dimensions by a factor of t scales area by a factor of t^2, so the top triangle has area t^2.

Push the small triangle on the left horizontally until it meets the small triangle on the right. Together, they form a triangle similar to the big triangle. And the base of the triangle that they form is $1-t$ times the base of the big triangle, so the triangle they form has area $(1-t)^2$.

Thus, altogether, the areas of triangles (a), (b), and (c) add up to

$$t^2 + (1-t)^2. \tag{8}$$

We want to choose t so as to maximize the shaded area. We can now easily express the shaded area as an explicit function of t: the shaded area is

$$1 - [t^2 + (1-t)^2],$$

which simplifies to $2t - 2t^2$. Maximizing this is quite easy. But instead we note that *maximizing* the shaded area is equivalent to *minimizing* the area covered by the three small triangles. We therefore need to choose t so that $t^2 + (1-t)^2$ is as small as possible.

We expand Expression (8), getting $2t^2 - 2t + 1$. "Complete the square." It is more convenient (fewer fractions) to rewrite the expression as $(1/2)(4t^2 - 4t + 2)$, which is equal to

$$\frac{1}{2}((2t-1)^2 + 1).$$

It is obvious that this reaches a minimum when $2t - 1 = 0$, and that the minimum *value* of the expression is $1/2$. It follows that the maximum possible shaded area is $1 - 1/2$.

(ii) Look now at the middle picture, and let the bottom shaded rectangle have base t times the bottom side of the big triangle. Then by the argument we gave above, the two bottom small triangles have combined area $(1-t)^2$. And the area of the triangle "above" the bottom shaded rectangle is t^2. So by the result of part (a), the minimum area of the region which is above the bottom shaded rectangle, but outside the top shaded rectangle, is $t^2/2$.

It follows that for this choice of t, the minimum area outside the two shaded rectangles is

$$(t-1)^2 + t^2/2.$$

We want to minimize this, that is, we want to minimize

$$(1/2)(3t^2 - 4t + 2).$$

By completing the square, we find that the minimum is reached when $t = 2/3$. Substitute. We find that the minimum area outside the shaded rectangles is $1/3$, and therefore the maximum area covered by the two shaded rectangles is $1 - 1/3$.

(iii) Look now at the right-hand picture, and let the bottom shaded rectangle have base t times the bottom side of the big triangle. Then by the argument we gave above, the two bottom small triangles have combined area $(1-t)^2$. The triangle "above" the bottom shaded rectangle has area t^2, and by the result of (ii), the smallest possible area of the region in this triangle but outside the top two shaded rectangles is $t^2/3$.

It follows that for this choice of t, the minimum area outside the three shaded rectangles is

$$(t-1)^2 + t^2/3.$$

We want to minimize this. Completing the square, we find that the minimum is reached at $t = 3/4$, and the minimum area turns out to be $1/4$. It follows that the maximum area covered by the three shaded rectangles is $1 - 1/4$.

Comment. "Completing the square" is a fundamental technique, variants of which come up in many branches of mathematics. Note that in the solution of (i), we used the fact that

$$t^2 + (1-t)^2 \qquad \text{is identically equal to} \qquad \frac{1}{2}((2t-1)^2 + 1).$$

The nice thing about $(2t-1)^2 + 1$ is that it is equal to $u^2 + 1$ where $u = 2t - 1$, meaning that it is symmetrical about $u = 0$, or equivalently, about $t = 1/2$. So completing the square reveals the hidden symmetry of a quadratic polynomial.

Comment. Imagine continuing the three pictures in the obvious way, with 4 rectangles, 5 rectangles, and so on. One might guess that with 4 rectangles, the maximum shaded area is $1 - 1/5$, that with 5 rectangles it is $1 - 1/6$, and so on. This is in fact true, and, for 4 rectangles, not hard to show, by imitating the argument of case (iii). One can in fact show in the same way that the maximum area when there are $n-1$ rectangles is $1 - 1/n$.

Another Way. We look only at the right-hand picture, that is, part (iii) of the problem, but by the time the analysis is completed, we will have dealt also with (i) and (ii), and much more.

Let the base of the big triangle be a. There are 4 horizontal lines in the picture. They divide the big triangle into 4 levels. The "top" level is a triangle. The other levels are in general trapezoids. (They are triangles if one of the base angles of the big triangle is a right angle. In what follows we will assume they are trapezoids. A minor modification of wording would deal with the case when they are triangles.)

Look at the four levels, say from bottom to top. The first level consists of a rectangle (shaded) plus two small triangles on the sides of the rectangle. Let the combined base of these two small triangles be x_1. Imagine sliding these triangles towards each other until they meet. We then get a triangle similar to the big triangle. The area of this triangle is therefore $(x_1/a)^2$.

Now look at the next level up. Again, this consists of a shaded rectangle and two small triangles. Let the combined base of these two triangles be x_2. Then the combined area of the two triangles is $(x_2/a)^2$.

Similarly, let the combined base of the two small triangles at the next level be x_3. Then the combined area of the two triangles at that level is $(x_3/a)^2$.

Finally, look at the upper level. This is a triangle, say with base x_4. The area of this triangle is $(x_4/a)^2$. The combined area of all the triangles mentioned is therefore $A(x_1, x_2, x_3, x_4)$, where

$$A(x_1, x_2, x_3, x_4) = (x_1^2 + x_2^2 + x_3^2 + x_4^2)/a^2. \tag{9}$$

To maximize the sum of the shaded areas, we must minimize $A(x_1, x_2, x_3, x_4)$. Note that the x_i are not independent of each other: we must have

$$x_1 + x_2 + x_3 + x_4 = a. \tag{10}$$

The minimization problem is solved by the following very useful standard lemma.

Lemma. Let a be a fixed number. Then the minimum value of

$$x_1^2 + x_2^2 + \cdots + x_n^2$$

subject to the condition

$$x_1 + x_2 + \cdots + x_n = a \tag{11}$$

is a^2/n, and is reached only when $x_1 = x_2 = \cdots = x_n = a/n$.

Proof. Let x_1, x_2, …, x_n be real numbers satisfying Equation (11). Note that

$$(nx_1 - a)^2 + (nx_2 - a)^2 + \cdots + (nx_n - a)^2 \geq 0, \tag{12}$$

since the left-hand side of (12) is a sum of squares. Moreover, this left-hand side can only be 0 if all the squares are 0, that is, if every x_i is equal to a/n.

Expand the left-hand side of (12). We get

$$n^2(x_1^2 + x_2^2 + \cdots + x_n^2) - 2na(x_1 + x_2 + \cdots + x_n) + na^2 \geq 0$$

and then, using the fact that $x_1 + x_2 + \cdots + x_n = a$, we get

$$n^2(x_1^2 + x_2^2 + \cdots + x_n^2) - na^2 \geq 0$$

or equivalently

$$x_1^2 + x_2^2 + \cdots + x_n^2 \geq a^2/n$$

with equality only when all the x_i are equal. □

Now we return to our geometric problem. Recall that to maximize the area of the shaded region in the rightmost picture, we needed to minimize the $F(x_1, x_2, x_3, x_4)$ of Equation (9) subject to the condition of Equation (10).

By Lemma , with $n = 4$, this minimum value is $1/4$. It follows that the maximum possible shaded area is $1 - 1/4$.

The same argument, with $n = 2$, and $n = 3$, takes care of parts (i) and (ii). Indeed the same argument takes care of the obvious generalization of questions (i), (ii), and (iii).

Problem 5. Consider the arithmetic sequence 8, 31, 54, 77, 100, and so on. The first term of this sequence is a perfect cube. Find three other perfect cubes in the sequence.

Solution. We can do a more or less simple search, perhaps computer-aided. Given the wording of the problem, this is perfectly legitimate, but perhaps not entirely satisfying. So we will take a more algebraic approach.

We would like to find positive integers n such that $8 + 23n$ is a perfect cube, say $8 + 23n = x^3$, where x is an integer. Equivalently, we want to find integers $x > 2$ such that $x^3 - 8$ is a multiple of 23.

This is relatively simple. For in general,

$$x^3 - a^3 = (x - a)(x^2 + ax + a^2)$$

and therefore

$$x^3 - 8 = (x - 2)(x^2 + 2x + 4).$$

So to make $x^3 - 8$ a multiple of 23, it is enough to make $x - 2$ a multiple of 23. Three possible values of x are therefore $x = 25$, 48, and 71.

Thus three other perfect cubes in our arithmetic sequence are 25^3, 48^3, and 71^3.

Comment. It is clear that in this way we can produce infinitely many perfect cubes in our arithmetic sequence.

Essentially the same argument shows that if n and d are positive integers, and the arithmetic sequence a, $a + d$, $a + 2d$, and so on contains a perfect n-th power, then it contains infinitely many perfect n-th powers.

Comment. In our solution, we showed that if x is of the shape $2 + 23m$, where m is a positive integer, then x^3 is a member of our arithmetic sequence. We now ask whether this is the only way to get perfect cubes in our sequence. Recall that we need to ensure that

$$(x - 2)(x^2 + 2x + 4)$$

is a multiple of 23. If this is the case, and $x - 2$ is not a multiple of 23, then $x^2 + 2x + 4$ must be. So we ask: can $x^2 + 2x + 4$ ever be a multiple of 23?

Note that

$$x^2 + 2x + 4 = (x + 1)^2 + 3 = y^2 + 3$$

where $y = x + 1$. So we ask: if y is an integer, can $y^2 + 3$ ever be a multiple of 23?

There are general procedures for examining this sort of question. For details, see almost any book on elementary number theory (there are some available on the Internet), or do a search, looking for the term "quadratic reciprocity." But we will proceed in a more primitive way.

Any integer y can be expressed in the form $y = 23s + t$, where s and t are integers and t ranges from 0 to 22, or, more conveniently, from -11 to 11. It follows that

$$y^2 + 3 = (23s)^2 + 46st + t^2 + 3,$$

and therefore $y^2 + 3$ is a multiple of 23 if and only if $t^2 + 3$ is a multiple of 23.

So need only examine $t^2 + 3$, for t ranging from -11 to 11. And since $(-t)^2 + 3 = t^2 + 3$, we need only examine values of t ranging from 0 to 11.

After this preliminary simplification, we compute $t^2 + 3$ for $t = 0$ to 11, and test for divisibility of the result by 23. Pretty quickly we find that none of the results is divisible by 23. So we have shown that the only numbers in our arithmetic sequence that are perfect cubes are the numbers $(2 + 23m)^3$, where m ranges over the non-negative integers.