Solutions to January 2007 Problems

Problem 1. Find three numbers in geometric progression whose sum is 2 and the sum of whose squares is 8.

Solution. It is natural to let the numbers be a, ar, and ar^2. That works just fine, but it is more attractive to let them be x, y, and z, with y the middle one. Since the numbers are in geometric progression, $xz = y^2$.

We were told that

$$x + y + z = 2 \qquad \text{and} \qquad x^2 + y^2 + z^2 = 8.$$

Note that the above equations are symmetric in x, y, and z. Roughly speaking, symmetry is good, an extra variable is bad. We have introduced an unnecessary variable for the sake of symmetry.

Square both sides of the first equation. We obtain

$$x^2 + y^2 + z^2 + 2(xy + xz + yz) = 4.$$

Now use the second equation to conclude that $xy+xz+yz = -2$. Since $xz = y^2$, we can rewrite this as $y(x + y + z) = -2$. It follows that $y = -1$.

Since $y = -1$, the first equation yields $x + z = 3$. Note that $xz = y^2 = 1$. Thus x and z are the roots of $u^2 - 3u + 1 = 0$. These roots are $(3 \pm \sqrt{5})/2$, and x can be either one of them, while z is the other.

Comment. The same problem, with sum 20 and sum of squares 140, can be found in Isaac Newton's *Universal Arithmetic* (late seventeenth century).

Problem 2. A tangent line to the ellipse $x^2 + 4y^2 = 4$ meets the x-axis at the point $(0, 2)$. What is the point of tangency? No calculus please!

Solution. Our ellipse can be viewed as the circle $x^2 + y^2 = 1$, scaled in the x-direction by a factor of 2. That is exactly how the drawing software drew it in Figure 1. Equivalently, the ellipse $x^2 + 4y^2 = 4$, scaled in the x-direction by a factor of $1/2$, becomes the circle $x^2 + y^2 = 1$. Imagine drawing a tangent line to $x^2 + 4y^2 = 4$ that passes through $(0, 2)$, and scaling in the x-direction by a factor of $1/2$. Then ellipse (on the right in Figure 1 transforms to circle (on the left in the same figure). Also, tangent line transforms to tangent line, and the point $(0, 2)$ transforms to $(0, 2)$.

Now we solve a simple "circle" problem: Find the point of tangency to the circle $x^2 + y^2 = 1$ of a tangent that passes through $(0, 2)$. It is obvious that there are two such points of tangency, symmetrical across the y-axis. For now let's concentrate on the one in the first quadrant.

Label things as in the diagram, with P the point of tangency. We get "lucky." Note that $\angle OPQ$ is a right angle, and that the sine of $\angle PQO$ is $1/2$. Thus this angle is a $30°$ angle, and therefore so is $\angle POX$. Since $OP = 1$, it follows easily that the coordinates of P are $(\sqrt{3}/2, 1/2)$. Even if we had not been "lucky"

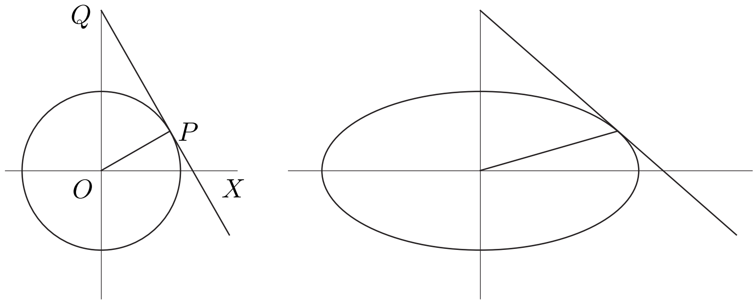


Figure 1: Circle and Ellipse

enough to get a 30° angle, we could have computed the point of tangency with not much difficulty by using properties of similar triangles.

Now go back to the ellipse by scaling by a factor of 2 in the x-direction. The point of tangency is transformed to $(\sqrt{3}, 1/2)$. Finally, recall that there are two possible points of tangency. The other is $(-\sqrt{3}, 1/2)$.

Comment. The sort of strategy used above is sometimes called "Transform, Solve, Transform Back." Use a transformation, in this case geometric, to transform a somewhat messy problem into one about a simpler situation. Solve the simpler problem, and transform back to solve the original problem. Large chunks of mathematics can be viewed as the carrying out of this sort of strategy.

Another Way. The above solution is (in my opinion) the "nice" one, but we can also grind things out. The lines that pass through the point $(0, 2)$ all have equations of the shape $y = mx + 2$, where m is the slope. (I lied: The vertical line through $(0, 2)$ has equation $x = 0$, but we are not interested in it.)

Compute where the line $y = mx+2$ meets the ellipse $x^2+4y^2 = 4$. Substitute for y in the equation of the ellipse. After a while we arrive at the equation

$$(4m^2 + 1)x^2 + 16mx + 12 = 0. \tag{1}$$

The tangent line meets the ellipse at precisely one point, meaning that Equation 1 has precisely one solution.

This is the case if and only if the discriminant is equal to 0, that is,

$$(16m)^2 - 4(4m^2 + 1)(12) = 0.$$

(We could bypass reference to the discriminant by completing the square.) Simplify, by dividing by 16 and collecting like terms. We get $4m^2 - 3 = 0$, which gives $m = \pm\sqrt{3}/2$. First look at $m = -\sqrt{3}/2$. Equation 1 becomes, after some simplification,

$$x^2 - 2\sqrt{3}x + 3 = 0,$$

which has $x = \sqrt{3}$ as its only solution. Substituting in $y = -(\sqrt{3}/2)x + 2$, we get $y = 1/2$. Similarly, the choice $m = \sqrt{3}/2$ gives $x = -\sqrt{3}$ and $y = 1/2$.

Problem 3. Thirteen people, including Alpha, Beta, Gamma, and Delta, are seated at random on the 13 chairs around a circular table. (a) What is the probability that no two of Alpha, Beta, and Gamma are immediate neighbours? (b) What is the probability that no two of Alpha, Beta, Gamma, and Delta are immediate neighbours?

Solution. (a) In looking at this problem, we can forget about Delta. We might as well assume that Alpha sits down first—she is the oldest. That leaves 12 empty chairs, of which we need to choose 2 for the two remaining people to sit at. Note that we will not be concerned about *which* of these two chairs Beta sits on.

The number of ways of choosing 2 chairs from the 12 is $\binom{12}{2}$, or, in high school notation, ${}_{12}\mathrm{C}_2$. It is a standard fact that $\binom{12}{2} = (12)(11)/2 = 66$. All ways of choosing the 2 chairs are equally likely.

Now we count for how many of these choices all 3 chosen chairs are separated by at least one chair. The 2 chairs on either side of Alpha are clearly forbidden. Let us number the 10 remaining chairs (say going to the right from Alpha's chair) 1, 2, 3, …, 10.

Suppose that chair 1 is the first of the 2 chosen chairs that one reaches, counting clockwise from Alpha. Then the remaining chair can be any one of 3, 4, 5, …, 10. So there are 8 possibilities that have chair 1 as the first one to the right of Alpha. Similarly, there are 7 possibilities that have chair 2 as the first occupied one to the right of Alpha, and so on, down to 1 possibility that has chair 8 as the first occupied chair to the right of Alpha. Thus the total number of ways of choosing the 2 chairs is $8 + 7 + \cdots + 1$, that is, 36. The required probability is therefore 36/66, or more simply 6/11. It is clear that the idea generalizes to 3 people and n chairs around a circular table.

(b) We can "recycle" the ideas of part (a), but the counting becomes somewhat more complicated. Let Alpha sit down first. Now there are 12 chairs left, of which 3 must be chosen. The choosing can be done in $\binom{12}{3}$ ways, all equally likely. A little computation shows that the number of ways is (12)(11)(1)/6, that is, 220.

Next we count the number of choices in which no two of the four chosen chairs are adjacent. The two chairs on either side of Alpha are forbidden, as before. Number the allowed chairs, starting with the second chair to the right of Alpha, with the numbers 1, 2, 3, …, 10.

Look at the chairs, rightward from Alpha. Maybe the first occupied chair is 6, in which case the only allowed possibilities for the next two are 8 and 10. So there is only 1 allowed pattern in which the first occupied chair to the right of Alpha is 6.

Or maybe the first occupied chair to the right of Alpha is 5. If the next one is 8, then the next must be 10 (1 possibility). If the next one is 7, then the next can be 9 or 10 (2 possibilies). Thus there are $1 + 2$ ways that the first occupied chair to the right of Alpha is 5.

Now suppose the first occupied chair to the right of Alpha is 4. There is 1 possibility if 8 is next, 2 possibilies if 7 is next, and 3 possibilities if 6 is next, for a total of $1+2+3$. Suppose next that the first occupied chair to the right of Alpha is 3. There is 1 possibility with 8 next, 2 possibilities with 7 next, 3 with 6 next, and 4 with 5 next, for a total of $1+2+3+4$. Similarly, there are $1+2+3+4+5$ possibilies with 2 the first occupied chair to the right of Alpha, and $1+2+3+4+5+6$ with 1 the first occupied chair to the right of Alpha.

Add up. We get a total of 56 possibilities. So the required probability is $56/220$, or more simply $14/55$. Generalization to n chairs is not difficult. The somewhat complicated sum we evaluated here can be tackled in general by the technique used to solve Problem 2 of the February 2007 problems. But in the simple numerical case done here, the summation was easily handled by brute force.

Another Way. (a) Just as in the first solution, let Alpha sit down first. That leaves 12 chairs, of which 2 can be chosen in $\binom{12}{2}$ equally likely ways. Now as before we must count the number of ways of choosing them in an allowed way (the ones next to Alpha must be empty, and the chosen ones must not be adjacent).

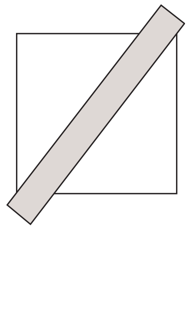
Imagine 10 white balls in a row, which stand for empty chairs. Then there are 9 "gaps" between white balls. We must choose 2 of these gaps to put black balls into (occupied chairs). This can be done in $\binom{9}{2}$ ways, so our probability is $\binom{9}{2}/\binom{12}{2}$, which turns out to be $6/11$.

(b) This is handled in exactly the same way! Imagine 9 white balls in a row, which stand for empty chairs. Then there are 8 "gaps" between white balls. We must choose 3 of these gaps to put black balls into (occupied chairs). This can be done in $\binom{8}{3}$ ways, so the required probability is $\binom{8}{3}/\binom{12}{3}$, which turns out to be $14/55$.

Comment. The idea of the previous approach easily generalizes. Suppose that we have $n+1$ chairs around a circular table, and we must seat $k+1$ people so that no two are next to each other. Let the oldest of the $k+1$ people sit first. Then if the remaining k sit at random, they can choose k chairs from n in $\binom{n}{k}$ equally likely ways.

Imagine now $n-k$ white balls in a row. There are $n-k-1$ "gaps" between them. We must choose k of these gaps to put black balls (occupied chairs). This can be done in $\binom{n-k-1}{k}$ ways.

Problem 4. A long ruler of width 1 is placed over an $a \times a$ square sheet of paper so that one edge of the ruler passes through a corner of the square, and the other edge passes through the opposite corner. Find the fraction of the area of the paper which is covered by the ruler.



Solution. The figure below is the diagram of the problem, except that the distracting shading has been removed, as have the parts of the ruler that stick out beyond the paper. Drop a perpendicular from X to the other edge of the ruler, meeting that edge at P. We can solve the problem by a straightforward

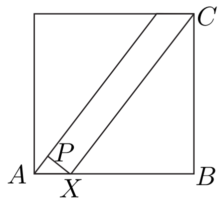


Figure 2: Solution to Ruler Problem

similar triangles argument. Note that triangles APX and XBC are similar. Let $x = AX$. Then $AP = \sqrt{x^2 - 1}$ and $XB = a - x$. It follows that

$$\frac{\sqrt{x^2-1}}{1} = \frac{a-x}{a}.$$

Square both sides and simplify. After a while we arrive at the equation

$$(ax^2 + x - 2a = 0,$$

which has the solution

$$x = \frac{-1+\sqrt{1+8a^2}}{2a}.$$

(We discarded the irrelevant negative solution of the quadratic equation.) Multiply by the "height" a of the parallelogram determined by the region of overlap, and divide by a^2 to find the required ratio. The result is

$$\frac{-1+\sqrt{1+8a^2}}{2a^2}.$$

We should add that the above analysis is only correct if $a \geq 1$. What happens if $0 < a < 1$? The diagram of the problem (and the analysis based on it) are then wrong, but the verbal statement of the problem makes sense for a while. A little playing with pictures shows that if the diagonal $a\sqrt{2}$ of the square is less than the width of the ruler, then the ruler cannot be placed as the problem specifies. But if $1/\sqrt{2} \leq a < 1$, then the ruler can be placed , and all of the paper is covered.

Problem 5. Note that the equation $x^2 - 6y^2 = 1$ has $x = 5$, $y = 2$ as a solution. Find a solution (x, y) of this equation, where x and y are integers and $x > 2007$. Hint: Maybe rewrite the equation as $(x - y\sqrt{6})(x + y\sqrt{6}) = 1$.

Solution. We follow the hint. Since $x = 5$, $y = 2$ is a solution, we have

$$(5 - 2\sqrt{6})(5 + 2\sqrt{6}) = 1. \tag{2}$$

Let n be a positive integer. Taking the n-th power of both sides of Equation 2, we obtain

$$(5-2\sqrt{6})^n(5+2\sqrt{6})^n = 1. \tag{3}$$

Imagine expanding $(5+2\sqrt{6})^n$, by multiplying out or by using the Binomial Theorem. The result is $A_n + B_n\sqrt{6}$, where A_n and B_n are integers. It is easy to see that the result of expanding $(5-2\sqrt{6})^n$ is then $A_n - B_n\sqrt{6}$. Thus Equation 3 can be rewritten as

$$(A_n - B_n\sqrt{6})(A_n + B_n\sqrt{6}) = 1, \tag{4}$$

or equivalently $A_n^2 - 6B_n^2 = 1$. It follows that the ordered pair (A_n, B_n) is a solution of the equation $x^2 - 6y^2 = 1$. As a matter of fact, *all* positive solutions (x, y) of the equation are of the form (A_n, B_n) for some positive integer n. This fact takes some effort to prove, and we do not need it to solve our problem.

Now we will find an $A_n > 2007$. One way to proceed is to note that

$$(5-2\sqrt{6})^n + (5+2\sqrt{6})^n = (A_n - B_n\sqrt{6}) + (A_n + B_n\sqrt{6}) = 2A_n.$$

But $5-2\sqrt{6}$ is about 0.1010205. In particular, if n is at all large, then $5-2\sqrt{6}$ is negligibly small, and $2A_n$ is "almost" equal to $(5+2\sqrt{6})^n$, but a little smaller.

So we search for an n such that $(5+2\sqrt{6})^n$ is bigger, say, than $2(2007)+1$. We can find such an n by using logarithms, or more simply by just playing with the calculator. Note that $5+2\sqrt{6}$ is about 9.8989795. Thus the smallest n that works is $n = 4$. The calculator gives that $(5+2\sqrt{6})^4$ is about 9601.9999. It follows that $2A_4 = 9602$, which gives $A_4 = 4801$. Parenthetically, since $A_4 = 4801$, the calculator gives $B_4 = 1960$, so we have obtained the solution $(4801, 1960)$ to the equation $x^2 - 6y^2 = 1$. Of course there are many larger solutions.

Another Way. An alternate approach to the computation uses A_n, B_n as above, but the computation is handled differently. Note that

$$\begin{aligned} A_{n+1} + B_{n+1} &= (5+2\sqrt{6})^{n+1} = (5+2\sqrt{6})(5+2\sqrt{6})^n \\ &= (5+2\sqrt{6})(A_n + B_n\sqrt{6}) \\ &= (5A_n + 12B_n) + (2A_n + 5B_n)\sqrt{6} \end{aligned}$$

and therefore

$$A_{n+1} = 5A_n + 12B_n, \qquad B_{n+1} = 2A_n + 5B_n. \tag{5}$$

Now we can compute. We have $A_1 = 5$, $B_1 = 2$. Thus by Equations 5, we have $A_2 = 49$, $B_2 = 20$. Applying the equations again, we get $A_3 = 485$, $B_3 = 198$. Next we get $A_4 = 4801$, and we are finished.

Another Way. We first note, as in previous solutions, that if $(5+2\sqrt{6})^n = A_n + B_n\sqrt{6}$, where A_n and B_n are integers, then (A_n, B_n) is a solution of $x^2 - 6y^2 = 1$.

Note that $5 \pm 2\sqrt{6}$ are the roots of the quadratic $x^2 - 10x + 1 = 0$. This is an easy consequence of a general result. Let α and β be any two numbers. Then α and β are the roots of the equation $x^2 - (\alpha + \beta)x + \alpha\beta = 0$.

Let $\theta = 5 + 2\sqrt{6}$. Then $\theta^2 - 10\theta + 1 = 0$, or equivalently $\theta^2 = 10\theta - 1$. Multiply both sides of this equation by θ^n. We obtain

$$\theta^{n+2} = 10\theta^{n+1} - \theta^n.$$

The equation above can be rewritten as

$$A_{n+2} + B_{n+2}\sqrt{6} = 10(A_{n+1} + B_{n+1}\sqrt{6}) - (A_n + B_n\sqrt{6}),$$

which yields the second order recurrence

$$A_{n+2} = 10A_{n+1} - A_n,$$

that reminds us of the recurrence that defines the Fibonacci sequence. (The B_n satisfy an essentially identical recurrence.)

Note that $A_0 = 1$ and $A_1 = 5$. Now we can compute easily. We have $A_0 = 1$ and $A_1 = 5$. Thus $A_2 = 49$, $A_3 = 485$, $A_4 = 4801$.

Comment. An equation of the type $x^2 - Dy^2 = 1$, where D is a non-square positive integer, and we are looking for solutions (x, y) in integers, is called a *Pell Equation.* (Some closely related equations are also sometimes called Pell equations.)

Pell was a minor 17th century English mathematician, rather better known as a spy. There is no good evidence that he ever had anything to do with the type of equation named after him—the attribution to Pell (due to Euler) is almost certainly a mistake.

A fairly detailed examination of Pell's Equation occurs in the work of the Indian mathematicians Brahmagupta (7th century) and Bhaskara (12th century). The equation should really be called Brahmagupta's Equation, but the name Pell has stuck.

The first European mathematicians to make a serious study of Pell's Equation were Fermat and Brouncker (17th century). The first complete analysis was by Lagrange (18th century). Pell's Equation comes up naturally in many places, and is connected with a number of important ideas in Number Theory, such as continued fractions.