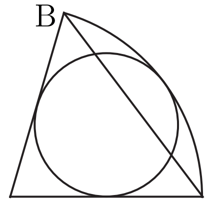
Problems, April 2007

Problem 1. Let $f(x) = x^2 + 2x - 1$. Solve the equation $f(f(x)) = f(x)$.

Problem 2. A circle is inscribed in a sector of a circle, as in the figure below. Suppose that the sector has radius R, the inscribed circle has radius r, and the chord AB has length $2c$. Show that

$$\frac{1}{r} = \frac{1}{c} + \frac{1}{R}.$$



Problem 3. Define the sequence c_0, c_1, c_2, and so on as follows: $c_0 = 2$, and for all non-negative integers n,

$$c_{n+1} = c_n^2 - c_n + 1.$$

(a) Suppose that $d > 1$ and $n > m$. Show that if d divides c_m, then c_n leaves a remainder of 1 when it is divided by d. (b) Use part (a) to show that there are infinitely many primes.

Problem 4. One can find 100 consecutive integers none of which is prime. For instance, all of the numbers $101!+2$, $101!+3$, $101!+4$, $\ldots$, $101!+101$ are composite. Show that there are 100 consecutive integers among which there are exactly 2 primes. (You will probably not find an *explicit* example—I haven't looked for one.)

Problem 5. We say that n has been partitioned into almost equal parts if n is expressed as

$$n = a_1 + a_2 + \cdots + a_k,$$

where the a_i are positive integers, $k \geq 1$, $a_1 \geq a_2 \geq \cdots \geq a_k$, and $a_k \geq a_1 - 1$. Examples of partitions of 8 into almost equal parts are 8, $4+4$, $3+3+2$, and $2+2+2+1+1$. How many partitions of n into almost equal parts are there?

http://www.pims.math.ca/education/math_problems/
http://www.math.ubc.ca/~adler/problems/