Problems, November 2007

Problem 1. It is not hard to verify with a calculator that

$$\frac{5^3+1^3}{5^3+4^3}=\frac{5+1}{5+4},\qquad \frac{67^3+41^3}{67^3+26^3}=\frac{67+41}{67+26},\qquad \frac{124^3+43^3}{124^3+81^3}=\frac{124+43}{124+81}.$$

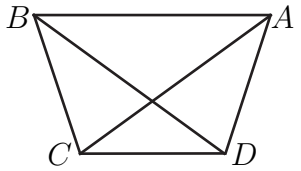
But it is easier to "cancel" the 3's from numerator and denominator! Explain why cancelling the 3's gives the right answer in these and similar cases.

Problem 2. Suppose that $0<x<y$. Which is larger,

$$\frac{1+x+x^2+\cdots+x^{100}}{1+x+x^2+\cdots+x^{101}} \quad \text{or} \quad \frac{1+y+y^2+\cdots+y^{100}}{1+y+y^2+\cdots+y^{101}}\,?$$

(This is straightforward only if it is approached the right way.)

Problem 3. In quadrilateral $ABCD$, $AB=AC=BD$ and $BC=CD=DA$. Find AB/BC.



Problem 4. At the beginning, bowl A contains 2 cups of sugar, and nothing else, and bowl B contains 1 cup of flour, and nothing else.

Xavier transfers 1 cup of stuff from A to B, mixes the contents of B thoroughly, then transfers 1 cup of stuff from B to A, and mixes the contents of A thoroughly. That's the end of the first *cycle.* Xavier then transfers 1 cup of stuff from A to B, mixes the contents of B thoroughly, transfers one cup of stuff from B to A, and mixes thoroughly. That's the end of the second cycle. Xavier continues. (a) How much flour is in bowl A at the end of the 4-th cycle? (b) Find a simple expression for the amount of flour in bowl A at the end of the n-th cycle, and prove that your expression is correct.

Problem 5. Alicia types all the numbers from 1 to 10^5 inclusive, and Beti types all the numbers from 1 to 10^6, inclusive. (a) Show that the total number of digits that Alicia types is equal to the total number of 0's that Beti types. (b) Generalize, with 10^k and 10^{k+1} in lieu of 10^5 and 10^6.

http://www.pims.math.ca/education/math_problems/
http://www.math.ubc.ca/~adler/problems/